Fock space representations of $U_q(\widehat{\mathfrak{sl}}_n)$

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Abstract

We give an introduction to the Fock space representations of the affine Lie algebras $\widehat{\mathfrak{sl}}_n$ and their quantum analogues $U_q(\widehat{\mathfrak{sl}}_n)$. We explain the construction of their canonical bases, and the relationship with decomposition matrices of q-Schur algebras at an nth root of 1. In the last section we give a brief survey of some recent higher level analogues of these constructions.

Nous donnons une introduction aux représentations de Fock des algèbres de Lie affines $\widehat{\mathfrak{sl}}_n$ et de leurs analogues quantiques $U_q(\widehat{\mathfrak{sl}}_n)$. Nous expliquons la construction de leurs bases canoniques, et leur relation avec les matrices de décomposition des q-algèbres de Schur en une racine n-ième de l'unité. Dans la dernière partie nous donnons un bref compte-rendu de résultats analogues récents pour les niveaux supérieurs à 1.

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Contents

1 Introduction

In the mathematical physics literature, the Fock space $\mathscr{F}$ is the carrier space of the natural irreducible representation of an infinite-dimensional Heisenberg Lie algebra $\mathfrak{H}$. Namely, $\mathscr{F}$ is the polynomial ring $\mathbb{C}[x_i \mid i \in \mathbb{N}^*]$, and $\mathfrak{H}$ is the Lie algebra generated by the derivations $\partial/\partial x_i$ and the operators of multiplication by x_i.

In the 70's it was realized that the Fock space could also give rise to interesting concrete realizations of highest weight representations of Kac-Moody affine Lie algebras $\widehat{\mathfrak{g}}$. Indeed $\widehat{\mathfrak{g}}$ has a natural Heisenberg subalgebra $\mathfrak{p}$ (the principal subalgebra) and the simplest highest weight $\widehat{\mathfrak{g}}$-module, called the basic representation of $\widehat{\mathfrak{g}}$, remains irreducible under restriction to $\mathfrak{p}$. Therefore, one can in principle extend the Fock space representation of $\mathfrak{p}$ to a Fock space representation $\mathscr{F}$ of $\widehat{\mathfrak{g}}$. This was first done for $\widehat{\mathfrak{g}} = \widehat{\mathfrak{sl}}_2$ by Lepowsky and Wilson [**LW**]. The Chevalley generators of $\widehat{\mathfrak{sl}}_2$ act on $\mathscr{F}$ via some interesting but complicated differential operators of infinite degree closely related to the vertex operators invented by physicists in the theory of dual resonance models. Soon after, this construction was generalized to all affine Lie algebras $\widehat{\mathfrak{g}}$ of A, D, E type [**KKLW**].

Independently and for different purposes (the theory of soliton equations) similar results were obtained by Date, Jimbo, Kashiwara and Miwa [**DJKM**] for classical affine Lie algebras. Their approach is however different. They first endow $\mathscr{F}$ with an action of an infinite rank affine Lie algebra and then restrict it to various subalgebras $\widehat{\mathfrak{g}}$ to obtain their basic representations. In type A for example, they realize in $\mathscr{F}$ the basic representation of $\mathfrak{gl}_\infty$ (related to the KP-hierarchy of soliton equations) and restrict it to natural subalgebras isomorphic to $\widehat{\mathfrak{sl}}_n$ $(n \geqslant 2)$ to get Fock space representations of these algebras (related to the KdV-hierarchy for $n = 2$). In this approach, the Fock space is rather the carrier space of the natural representation of an infinite-dimensional Clifford algebra, that is, an infinite dimensional analogue of an exterior algebra. The natural isomorphism between this "fermionic" construction and the previous "bosonic" construction is called the boson-fermion correspondence.

The basic representation of $\widehat{\mathfrak{g}}$ has level one. Higher level irreducible representations can also be constructed as subrepresentations of higher level Fock space representations of $\widehat{\mathfrak{g}}$ [**F1**, **F2**].

After quantum enveloping algebras of Kac-Moody algebras were invented by Jimbo and Drinfeld, it became a natural question to construct the q-analogues of the above Fock space representations. The first results in this direction were obtained by Hayashi [**H**]. His construction was soon developed by Misra and Miwa [**MM**], who showed that the Fock space representation of $U_q(\widehat{\mathfrak{sl}}_n)$ has a crystal basis (crystal bases had just been introduced by Kashiwara) and described it completely in terms of Young diagrams. This was the first example of a crystal basis of an infinite-dimensional representation. Another construction of the level one Fock space representation of $U_q(\widehat{\mathfrak{sl}}_n)$ was given by Kashiwara, Miwa and Stern [**KMS**], in terms of semi-infinite q-wedges. This relied on the polynomial tensor representations of $U_q(\widehat{\mathfrak{sl}}_n)$ which give rise to the quantum affine analogue of the Schur-Weyl duality obtained by Ginzburg, Reshetikhin and Vasserot [**GRV**], and Chari and Pressley [**CP**] independently.

In [**LLT**] and [**LT1**], some conjectures were formulated relating the decomposition matrices of type A Hecke algebras and q-Schur algebras at an nth root of unity on the one hand, and the global crystal basis of the Fock space representation $\mathscr{F}$ of $U_q(\widehat{\mathfrak{sl}}_n)$ on the other hand. Note that [**LT1**] contains in particular the definition of the global basis of $\mathscr{F}$, which does not follow from the general theory of Kashiwara or Lusztig. The conjecture on Hecke algebras was proved by Ariki [**A1**], and the conjecture on Schur algebras by Varagnolo and Vasserot [**VV**].

Slightly after, Uglov gave a remarkable generalization of the results of [**KMS**], [**LT1**], and [**VV**] to higher levels. Together with Takemura [**TU**], he introduced a semi-infinite wedge realization of the level ℓ Fock space representations of $U_q(\widehat{\mathfrak{sl}}_n)$, and in [**U1**, **U2**] he constructed their canonical bases and expressed their coefficients in terms of Kazhdan-Lusztig polynomials for the affine symmetric groups.

A full understanding of these coefficients as decomposition numbers is still missing. Recently, Yvonne [**Y2**] has formulated a precise conjecture stating that, under certain conditions on the components of the multi-charge of the Fock space, the coefficients of Uglov's bases should give the decomposition numbers of the cyclotomic q-Schur algebras of Dipper, James and Mathas [**DiJaMa**]. Rouquier [**R**] has generalized this conjecture to all multi-charges. In his version the cyclotomic q-Schur algebras are replaced by some quasi-hereditary algebras arising from the category $\mathscr{O}$ of the rational Cherednik algebras attached to complex reflection groups of type $G(\ell,1,m)$.

In these lectures we first present in Section 2 the Fock space representations of the affine Lie algebra $\widehat{\mathfrak{sl}}_n$. We chose the most suitable construction for our purpose of q-deformation, namely, we realize $\mathscr{F}$ as a space of semi-infinite wedges (the fermionic picture). In Section 3 we explain the level one Fock space representation of $U_q(\widehat{\mathfrak{sl}}_n)$ and construct its canonical bases. In Section 4 we explain the conjecture of [**LT1**] and its proof by Varagnolo and Vasserot. Finally, in Section 5 we indicate the main lines of Uglov's construction of higher level Fock space representations of $U_q(\widehat{\mathfrak{sl}}_n)$, and of their canonical bases, and we give a short review of Yvonne's work.

2 Fock space representations of $\widehat{\mathfrak{sl}}_n$

2.1 The Lie algebra $\widehat{\mathfrak{sl}}_n$ and its wedge space representations

We fix an integer $n \geqslant 2$.

2.1.1 The Lie algebra $\mathfrak{sl}_n$

The Lie algebra $\mathfrak{g} = \mathfrak{sl}_n$ of traceless $n \times n$ complex matrices has Chevalley generators

$$E_i = E_{i,i+1}, \quad F_i = E_{i+1,i}, \quad H_i = E_{ii} - E_{i+1,i+1}, \qquad (1 \leqslant i \leqslant n-1).$$

Its natural action on $V = \mathbb{C}^n = \oplus_{i=1}^n \mathbb{C} v_i$ is

$$E_i v_j = \delta_{j,i+1} v_i, \quad F_i v_j = \delta_{j,i} v_{i+1}, \quad H_i v_j = \delta_{j,i} v_i - \delta_{j,i+1} v_{i+1}, \qquad (1 \leqslant i \leqslant n-1).$$

We may picture the action of $\mathfrak{g}$ on V as follows

$$v_1 \xrightarrow{F_1} v_2 \xrightarrow{F_2} \cdots \xrightarrow{F_{n-1}} v_n$$

2.1.2 The Lie algebra $L(\mathfrak{sl}_n)$

The loop space $L(\mathfrak{g}) = \mathfrak{g} \otimes \mathbb{C}[z, z^{-1}]$ is a Lie algebra under the Lie bracket

$$[a \otimes z^k, b \otimes z^l] = [a,b] \otimes z^{k+l}, \qquad (a,b \in \mathfrak{g},\ k,l \in \mathbb{Z}).$$

The loop algebra $L(\mathfrak{g})$ naturally acts on $V(z) = V \otimes \mathbb{C}[z, z^{-1}]$ by

$$(a \otimes z^k) \cdot (v \otimes z^l) = av \otimes z^{k+l}, \qquad (a \in \mathfrak{g},\ v \in V,\ k,l \in \mathbb{Z}).$$

2.1.3 The Lie algebra $\widehat{\mathfrak{sl}}_n$

The affine Lie algebra $\widehat{\mathfrak{g}} = \widehat{\mathfrak{sl}}_n$ is the central extension $L(\mathfrak{g}) \oplus \mathbb{C}c$ with Lie bracket

$$[a \otimes z^k + \lambda c,\ b \otimes z^l + \mu c] = [a,b] \otimes z^{k+l} + k\,\delta_{k,-l}\,\mathrm{tr}\,(ab)\,c, \qquad (a,b \in \mathfrak{g},\ \lambda,\mu \in \mathbb{C},\ k,l \in \mathbb{Z}).$$

This is a Kac-Moody algebra of type $A_{n-1}^{(1)}$ with Chevalley generators

$$e_i = E_i \otimes 1, \quad f_i = F_i \otimes 1, \quad h_i = H_i \otimes 1, \qquad (1 \leqslant i \leqslant n-1),$$

$$e_0 = E_{n1} \otimes z, \quad f_0 = E_{1n} \otimes z^{-1}, \quad h_0 = (E_{nn} - E_{11}) \otimes 1 + c.$$

We denote by Λ_i $(i = 0, 1, \ldots, n-1)$ the fundamental weights of $\widehat{\mathfrak{g}}$. By definition, they satisfy

$$\Lambda_i(h_j) = \delta_{ij}, \qquad (0 \leqslant i,j \leqslant n-1).$$

Let $V(\Lambda)$ be the irreducible $\widehat{\mathfrak{g}}$-module with highest weight Λ [**K**, §9.10]. If $\Lambda = \sum_i a_i \Lambda_i$ then the central element $c = \sum_i h_i$ acts as $\sum_i a_i \mathrm{Id}$ on $V(\Lambda)$, and we call $\ell = \sum_i a_i$ the level of $V(\Lambda)$. More generally, a representation V of $\widehat{\mathfrak{g}}$ is said to have *level* ℓ if c acts on V by multiplication by ℓ.

The loop representation $V(z)$ can also be regarded as a representation of $\widehat{\mathfrak{g}}$, in which c acts trivially. Define

$$u_{i-nk} = v_i \otimes z^k, \qquad (1 \leqslant i \leqslant n,\ k \in \mathbb{Z}).$$

Then $(u_j \mid j \in \mathbb{Z})$ is a $\mathbb{C}$-basis of $V(z)$. We may picture the action of $\widehat{\mathfrak{g}}$ on $V(z)$ as follows

$$\cdots \xrightarrow{f_{n-2}} u_{-1} \xrightarrow{f_{n-1}} u_0 \xrightarrow{f_0} u_1 \xrightarrow{f_1} u_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} u_n \xrightarrow{f_0} u_{n+1} \xrightarrow{f_1} u_{n+2} \xrightarrow{f_2} \cdots$$

Note that this is not a highest weight representation.

2.1.4 The tensor representations

For $r \in \mathbb{N}^*$, we consider the tensor space $V(z)^{\otimes r}$. The Lie algebra $\widehat{\mathfrak{g}}$ acts by derivations on the tensor algebra of $V(z)$. This induces an action on each tensor power $V(z)^{\otimes r}$, namely,

$$x(u_{i_1} \otimes \cdots \otimes u_{i_r}) = (xu_{i_1}) \otimes \cdots \otimes u_{i_r} + \cdots + u_{i_1} \otimes \cdots \otimes (xu_{i_r}), \qquad (x \in \widehat{\mathfrak{g}},\ i_1, \cdots, i_r \in \mathbb{Z}).$$

Again c acts trivially on $V(z)^{\otimes r}$.

We have a vector space isomorphism $V^{\otimes r} \otimes \mathbb{C}[z_1^{\pm}, \ldots, z_r^{\pm}] \xrightarrow{\sim} V(z)^{\otimes r}$ given by

$$(v_{i_1} \otimes \cdots \otimes v_{i_r}) \otimes z_1^{j_1} \cdots z_r^{j_r} \mapsto (v_{i_1} \otimes z^{j_1}) \otimes \cdots \otimes (v_{i_r} \otimes z^{j_r}), \qquad (1 \leqslant i_1, \ldots, i_r \leqslant n,\ j_1, \ldots, j_r \in \mathbb{Z}).$$

2.1.5 Action of the affine symmetric group

The symmetric group $\mathfrak{S}_r$ acts on $V^{\otimes r}\otimes\mathbb{C}[z_1^{\pm},\ldots,z_r^{\pm}]$ by

$$\sigma(v_{i_1}\otimes\cdots\otimes v_{i_r})\otimes z_1^{j_1}\cdots z_r^{j_r}=(v_{i_{\sigma^{-1}(1)}}\otimes\cdots\otimes v_{i_{\sigma^{-1}(r)}})\otimes z_1^{j_{\sigma^{-1}(1)}}\cdots z_r^{j_{\sigma^{-1}(r)}},\qquad(\sigma\in\mathfrak{S}_r).$$

Moreover the abelian group $\mathbb{Z}^r$ acts on this space, namely $(k_1,\ldots,k_r)\in\mathbb{Z}^r$ acts by multiplication by $z_1^{k_1}\cdots z_r^{k_r}$. Hence we get an action on $V(z)^{\otimes r}$ of the affine symmetric group $\widehat{\mathfrak{S}}_r:=\mathfrak{S}_r\ltimes\mathbb{Z}^r$. Clearly, this action commutes with the action of $\widehat{\mathfrak{g}}$.

It is convenient to describe this action in terms of the basis

$$(u_{\mathbf{i}}=u_{i_1}\otimes\ \cdots\otimes u_{i_r}\mid\mathbf{i}=(i_1,\ldots,i_r)\in\mathbb{Z}^r).$$

Denote by $s_k=(k,k+1),\ (k=1,\ldots,r-1)$ the simple transpositions of $\mathfrak{S}_r$. The affine symmetric group $\widehat{\mathfrak{S}}_r$ acts naturally on $\mathbb{Z}^r$ via

$$\begin{aligned}s_k\cdot\mathbf{i}&=(i_1,\ldots,i_{k+1},i_k,\ldots i_r),\quad(1\leqslant k\leqslant r-1),\\ z_j\cdot\mathbf{i}&=(i_1,\ldots,i_j-n,\ldots,i_r),\quad(1\leqslant j\leqslant r),\end{aligned}$$

and we have

$$w\,u_{\mathbf{i}}=u_{w\cdot\mathbf{i}},\qquad(\mathbf{i}\in\mathbb{Z}^r,\ w\in\widehat{\mathfrak{S}}_r).$$

Thus the basis vectors $u_{\mathbf{i}}$ are permuted, and each orbit has a unique representative $u_{\mathbf{i}}$ with

$$\mathbf{i}\in A_r:=\{\mathbf{i}\in\mathbb{Z}^r\mid 1\leqslant i_1\leqslant i_2\leqslant\cdots\leqslant i_r\leqslant n\}.$$

Let $\mathbf{i}\in A_r$. The stabilizer $\mathfrak{S}_{\mathbf{i}}$ of $u_{\mathbf{i}}$ in $\widehat{\mathfrak{S}}_r$ is the subgroup of $\mathfrak{S}_r$ generated by the s_k such that $i_k=i_{k+1}$, a parabolic subgroup. Hence we have finitely many orbits, and each of them is of the form $\widehat{\mathfrak{S}}_r/\mathfrak{S}_{\mathbf{i}}$ for some parabolic subgroup $\mathfrak{S}_{\mathbf{i}}$ of $\mathfrak{S}_r$.

2.1.6 The wedge representations

Consider now the wedge product $\wedge^rV(z)$. It has a basis consisting of normally ordered wedges

$$\wedge u_{\mathbf{i}}:=u_{i_1}\wedge u_{i_2}\wedge\cdots\wedge u_{i_r},\qquad(\mathbf{i}=(i_1>i_2>\cdots>i_r)\in\mathbb{Z}^r).$$

The Lie algebra $\widehat{\mathfrak{g}}$ also acts by derivations on the exterior algebra $\wedge V(z)$, and this restricts to an action on $\wedge^rV(z)$, namely,

$$x(u_{i_1}\wedge\cdots\wedge u_{i_r})=(xu_{i_1})\wedge\cdots\wedge u_{i_r}+\cdots+u_{i_1}\wedge\cdots\wedge(xu_{i_r}),\qquad(x\in\widehat{\mathfrak{g}},\ i_1,\cdots,i_r\in\mathbb{Z}).$$

Again c acts trivially on $\wedge^rV(z)$.

We may think of $\wedge^rV(z)$ as the vector space quotient of $V(z)^{\otimes r}$ by the subspace

$$\mathscr{I}_r:=\sum_{k=1}^{r-1}\mathrm{Im}\,(s_k+\mathrm{Id})\subset V(z)^{\otimes r}$$

of "partially symmetric tensors".

2.1.7 Action of the center of $\mathbb{C}\widehat{\mathfrak{S}}_r$

The space $\mathscr{I}_r$ is not stable under the action of $\widehat{\mathfrak{S}}_r$. (For example if $r=2$, we have $z_1(u_0\otimes u_0)=u_{-n}\otimes u_0$, which is not symmetric.) Hence $\widehat{\mathfrak{S}}_r$ does not act on the wedge product $\wedge^r V(z)$. However, for every $k\in\mathbb{Z}$, the element

$$b_k := \sum_{i=1}^{r} z_i^k$$

of the group algebra $\mathbb{C}\widehat{\mathfrak{S}}_r$ commutes with $\mathfrak{S}_r\subset\mathbb{Z}\widehat{\mathfrak{S}}_r$. Therefore it has a well-defined action on $\wedge^r V(z)$, given by

$$b_k(\wedge u_{\mathbf{i}}) = u_{i_1-nk}\wedge u_{i_2}\wedge\cdots\wedge u_{i_r} + u_{i_1}\wedge u_{i_2-nk}\wedge\cdots\wedge u_{i_r}+\cdots+u_{i_1}\wedge u_{i_2}\wedge\cdots\wedge u_{i_r-nk}.$$

This action commutes with the action of $\widehat{\mathfrak{g}}$ on $\wedge^r V(z)$. The elements b_k $(k\in\mathbb{Z})$ generate a subalgebra of $\mathbb{C}\widehat{\mathfrak{S}}_r$ isomorphic to the algebra of symmetric Laurent polynomials in r variables.

2.2 The level one Fock space representation of $\widehat{\mathfrak{sl}}_n$

We want to pass to the limit $r\to\infty$ in the wedge product $\wedge^r V(z)$.

2.2.1 The Fock space $\mathscr{F}$

For $s\geqslant r$ define a linear map $\varphi_{r,s}:\wedge^r V(z)\longrightarrow\wedge^s V(z)$ by

$$\varphi_{r,s}(\wedge u_{\mathbf{i}}) := \wedge u_{\mathbf{i}}\wedge u_{-r}\wedge u_{-r-1}\wedge\cdots\wedge u_{-s+1}.$$

Then clearly $\varphi_{s,t}\circ\varphi_{r,s}=\varphi_{r,t}$ for any $r\leqslant s\leqslant t$. Let

$$\wedge^\infty V(z) := \lim_{\to}\wedge^r V(z)$$

be the direct limit of the vector spaces $\wedge^r V(z)$ with respect to the maps $\varphi_{r,s}$. Each $\wedge u_{\mathbf{i}}$ in $\wedge^r V(z)$ has an image $\varphi_r(\wedge u_{\mathbf{i}})\in\wedge^\infty V(z)$, which should be thought of as the "infinite wedge"

$$\varphi_r(\wedge u_{\mathbf{i}}) \equiv \wedge u_{\mathbf{i}}\wedge u_{-r}\wedge u_{-r-1}\wedge\cdots\wedge u_{-s}\wedge\cdots$$

The space $\mathscr{F}:=\wedge^\infty V(z)$ is called the *fermionic Fock space*. It has a basis consisting of all infinite wedges

$$u_{i_1}\wedge u_{i_2}\wedge\cdots\wedge u_{i_r}\wedge\cdots,\qquad (i_1>i_2>\cdots>i_r>\cdots),$$

which coincide except for finitely many indices with the special infinite wedge

$$|\emptyset\rangle := u_0\wedge u_{-1}\wedge\cdots\wedge u_{-r}\wedge u_{-r-1}\wedge\cdots$$

called the *vacuum vector*. It is convenient to label this basis by partitions. A partition λ is a finite weakly decreasing sequence of positive integers $\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_s>0$. We make it into an infinite sequence by putting $\lambda_j=0$ for $j>s$, and we set

$$|\lambda\rangle := u_{i_1}\wedge u_{i_2}\wedge\cdots\wedge u_{i_r}\wedge\cdots,$$

where $i_k=\lambda_k-k+1$ $(k\geqslant 1)$.

This leads to a natural $\mathbb{N}$-grading on $\mathscr{F}$ given by $\deg|\lambda\rangle=\sum_i\lambda_i$ for every partition λ. The dimension of the degree d component $\mathscr{F}^{(d)}$ of $\mathscr{F}$ is equal to number of partitions λ of d. This can be encoded in the following generating function

$$\sum_{d\geqslant 0}\dim\mathscr{F}^{(d)}t^d=\prod_{k\geqslant 1}\frac{1}{1-t^k}. \tag{1}$$

2.2.2 Young diagrams

Let $\mathscr{P}$ denote the set of all partitions. Elements of $\mathscr{P}$ are represented graphically by Young diagrams. For example the partition $\lambda = (3,2)$ is represented by

$$\begin{array}{|c|c|c}\cline{1-2} & & \\ \hline & & \multicolumn{1}{c|}{} \\ \hline\end{array}$$

If γ is the cell in column number i and row number j, we call $i-j$ the *content* of γ. For example the contents of the cells of λ are

$$\begin{array}{|c|c|c}\cline{1-2} -1 & 0 & \\ \hline 0 & 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array}$$

Given $i \in \{0,\dots,n-1\}$, we say that γ is an *i-cell* if its content $c(\gamma)$ is congruent to i modulo n.

2.2.3 The total Fock space $\mathbb{F}$

It is sometimes convenient to consider, for $m \in \mathbb{Z}$, similar Fock spaces $\mathscr{F}_m$ obtained by using the modified family of embeddings

$$\varphi_{r,s}^{(m)}(\wedge u_{\mathbf{i}}) := \wedge u_{\mathbf{i}} \wedge u_{m-r} \wedge u_{m-r-1} \wedge \cdots \wedge u_{m-s+1}.$$

The space $\mathscr{F}_m$ has a basis consisting of all infinite wedges

$$u_{i_1} \wedge u_{i_2} \wedge \cdots \wedge u_{i_r} \wedge \cdots, \qquad (i_1 > i_2 > \cdots > i_r > \cdots),$$

which coincide, except for finitely many indices, with

$$|\emptyset_m\rangle := u_m \wedge u_{m-1} \wedge \cdots \wedge u_{m-r} \wedge u_{m-r-1} \wedge \cdots$$

The space $\mathbb{F} = \oplus_{m\in\mathbb{Z}}\mathscr{F}_m$ is called the *total Fock space*, and $\mathscr{F}_m$ is the *Fock space of charge m*. Most of the time, we shall only deal with $\mathscr{F} = \mathscr{F}_0$.

2.2.4 Action of f_i

Consider the action of the Chevalley generators f_i $(i = 0,1,\dots n-1)$ of $\widehat{\mathfrak{g}}$ on the sequence of vector spaces $\wedge^r V(z)$. It is easy to see that, for $\wedge u_{\mathbf{i}} \in \wedge^r V(z)$,

$$f_i\, \varphi_{r,s}(\wedge u_{\mathbf{i}}) = \varphi_{r+1,s}\, f_i\, \varphi_{r,r+1}(\wedge u_{\mathbf{i}})$$

for all $s > r$.

Exercise 1 Check this.

Hence one can define an endomorphism f_i^∞ of $\mathscr{F}$ by

$$f_i^\infty\, \varphi_r(\wedge u_{\mathbf{i}}) = \varphi_{r+1}\, f_i\, \varphi_{r,r+1}(\wedge u_{\mathbf{i}}). \tag{2}$$

The action of f_i^∞ on the basis $\{|\lambda\rangle \mid \lambda \in \mathscr{P}\}$ has the following combinatorial translation in terms of Young diagrams:

$$f_i^\infty|\lambda\rangle = \sum_\mu |\mu\rangle, \tag{3}$$

where the sum is over all partitions μ obtained from λ by adding an i-cell.

Exercise 2 Check it. Check that, for $n = 2$, we have

$$f_0^\infty|3,1\rangle = |3,2\rangle + |3,1,1\rangle, \qquad f_1^\infty|3,1\rangle = |4,1\rangle.$$

2.2.5 Action of e_i

Similarly, one can check that putting $e_i^\infty \varphi_r(\wedge u_{\mathbf{i}}) := \varphi_r(e_i \wedge u_{\mathbf{i}})$ one gets a well-defined endomorphism of $\mathscr{F}$. Its combinatorial description is given by

$$e_i^\infty |\mu\rangle = \sum_\lambda |\lambda\rangle, \tag{4}$$

where the sum is over all partitions λ obtained from μ by removing an i-cell.

Exercise 3 Check it. Check that, for $n = 2$, we have

$$e_0^\infty |3,1\rangle = |2,1\rangle, \qquad e_1^\infty |3,1\rangle = |3\rangle.$$

Theorem 1 *The map*

$$e_i \mapsto e_i^\infty, \quad f_i \mapsto f_i^\infty, \qquad (0 \leqslant i \leqslant n-1),$$

extends to a level one representation of $\widehat{\mathfrak{g}}$ on $\mathscr{F}$.

Proof — Let μ be obtained from λ by adding an i-cell γ. Then we call γ a *removable i-cell* of μ or an *addable i-cell* of λ. One first deduces from Eq. (3) (4) that $h_i^\infty := [e_i^\infty, f_i^\infty]$ is given by

$$h_i^\infty |\lambda\rangle = N_i(\lambda)|\lambda\rangle, \tag{5}$$

where $N_i(\lambda)$ is the number of addable i-cells of λ minus the number of removable i-cells of λ. It is then easy to check from Eq. (3) (4) (5) that the endomorphisms $e_i^\infty,\ f_i^\infty,\ h_i^\infty$ $(0 \leqslant i \leqslant n-1)$ satisfy the Serre relations of the Kac-Moody algebra of type $A_{n-1}^{(1)}$ (see [**K**, §0.3, §9.11]). Now, $c = h_0 + h_1 + \cdots + h_{n-1}$ acts by

$$c|\lambda\rangle = \sum_{i=0}^{n-1} N_i(\lambda)|\lambda\rangle.$$

Clearly, the difference between the total number of addable cells and the total number of removable cells of any Young diagram is always equal to 1, hence we have $\sum_{i=0}^{n-1} N_i(\lambda) = 1$ for every λ. □

Thus the Fock space $\mathscr{F}$ is endowed with an action of $\widehat{\mathfrak{g}}$. Note that, although every $\wedge^r V(z)$ is a level 0 representation, their limit $\mathscr{F}$ is a level 1 representation. This representation is called the *level 1 Fock space representation* of $\widehat{\mathfrak{g}}$.

Note also that $\wedge^r V(z)$ has no primitive vector, *i.e.* no vector killed by every e_i. But $\mathscr{F}$ has many primitive vectors.

Exercise 4 Take $n = 2$. Check that $v_0 = |\emptyset\rangle$ and $v_1 = |2\rangle - |1,1\rangle$ are primitive vectors. Can you find an infinite family of primitive vectors ?

2.2.6 Action of b_k

Recall the endomorphisms b_k $(k \in \mathbb{Z})$ of $\wedge^r V(z)$ (see §2.1.7). It is easy to see that, if $k \neq 0$, the vector $\varphi_s\, b_k\, \varphi_{r,s}(\wedge u_{\mathbf{i}})$ is independent of s for $s > r$ large enough. Hence, one can define endomorphisms b_k^∞ $(k \in \mathbb{Z}^*)$ of $\mathscr{F}$ by

$$b_k^\infty \varphi_r(\wedge u_{\mathbf{i}}) := \varphi_s\, b_k\, \varphi_{r,s}(\wedge u_{\mathbf{i}}) \qquad (s \gg 1). \tag{6}$$

In other words

$$b_k^\infty (u_{i_1} \wedge u_{i_2} \wedge \cdots) = (u_{i_1 - nk} \wedge u_{i_2} \wedge \cdots) + (u_{i_1} \wedge u_{i_2 - nk} \wedge \cdots) + \cdots$$

where in the right-hand side, only finitely many terms are nonzero. By construction, these endomorphisms commute with the action of $\widehat{\mathfrak{g}}$ on $\mathscr{F}$. However they no longer generate a commutative algebra but a Heisenberg algebra, as we shall now see.

2.2.7 Bosons

In fact, we will consider more generally the endomorphisms β_k $(k \in \mathbb{Z}^*)$ of $\mathscr{F}$ defined by

$$\beta_k(u_{i_1} \wedge u_{i_2} \wedge \cdots) = (u_{i_1-k} \wedge u_{i_2} \wedge \cdots) + (u_{i_1} \wedge u_{i_2-k} \wedge \cdots) + \cdots \tag{7}$$

so that $b_k^\infty = \beta_{nk}$.

Proposition 1 *For $k, l \in \mathbb{Z}^*$, we have $[\beta_k, \beta_l] = \delta_{k,-l}\, k\, \mathrm{Id}_{\mathscr{F}}$.*

Proof — Recall the total Fock space $\mathbb{F} = \oplus_{m\in\mathbb{Z}}\mathscr{F}_m$ introduced in §2.2.3. Clearly, the definition of the endomorphism β_k of $\mathscr{F}$ given by Eq. (7) can be extended to any $\mathscr{F}_m$. So we may consider β_k as an endomorphism of $\mathbb{F}$ preserving each $\mathscr{F}_m$. For $i \in \mathbb{Z}$, we also have the endomorphism w_i of $\mathbb{F}$ defined by $w_i(v) = u_i \wedge v$. It sends $\mathscr{F}_m$ to $\mathscr{F}_{m+1}$ for every $m \in \mathbb{Z}$. For any $v \in \mathbb{F}$, we have

$$\beta_k w_i(v) = \beta_k(u_i \wedge v) = u_{i-k} \wedge v + u_i \wedge \beta_k(v),$$

hence $[\beta_k, w_i] = w_{i-k}$. It now follows from the Jacobi identity that

$$[[\beta_k, \beta_l], w_i] = [[\beta_k, w_i], \beta_l] - [[\beta_l, w_i], \beta_k] = [w_{i-k}, \beta_l] - [w_{i-l}, \beta_k] = -w_{i-k-l} + w_{i-l-k} = 0.$$

Let us write for short $\gamma = [\beta_k, \beta_l]$. We first want to show that $\gamma = \kappa\,\mathrm{Id}_{\mathbb{F}}$ for some constant κ. Consider a basis element $v \in \mathscr{F}_m$. By construction we have

$$v = u_{i_1} \wedge \cdots \wedge u_{i_N} \wedge |\emptyset_{m-N}\rangle$$

for some large enough N and $i_1 > i_2 > \cdots > i_N > m-N$. Since γ commutes with every w_i, we have

$$\gamma(v) = u_{i_1} \wedge \cdots \wedge u_{i_N} \wedge \gamma(|\emptyset_{m-N}\rangle). \tag{8}$$

We have

$$\gamma(|\emptyset_{m-N}\rangle) = \sum_j \alpha_j\, u_{m_{1,j}} \wedge \cdots \wedge u_{m_{L,j}} \wedge |\emptyset_{m-N-L}\rangle$$

for some large enough L and $m_{1,j} > \cdots > m_{L,j} > m-N-L$. Clearly, $m_{1,j} \leqslant m-N+|k|+|l|$. Taking N large enough, we can assume that $\{m-N+1, m-N+2, \ldots, m-N+|k|+|l|\} \subset \{i_1, \ldots, i_N\}$. Hence we must have $m_{1,j} = m-N$ for every j, and this forces $m_{2,j} = m-N-1, \ldots, m_{L,j} = m-N-L+1$. This shows that $\gamma(v) = \kappa_v\, v$ for some scalar κ_v. Now Eq. (8) shows that $\kappa_v = \kappa_{|\emptyset_{m-N}\rangle}$ for N large enough, so $\kappa_v = \kappa$ does not depend on v.

Finally, we have to calculate κ. For that, we remark that β_k is an endomorphism of degree $-k$, for the grading of $\mathscr{F}$ defined in §2.2.1. Hence we see that if $k+l \neq 0$ then $\kappa = 0$. So suppose $k = -l > 0$, and let us calculate $[\beta_k, \beta_l]|\emptyset\rangle$. For degree reasons this reduces to $\beta_k\beta_{-k}|\emptyset\rangle$. It is easy to see that $\beta_{-k}|\emptyset\rangle$ is a sum of k terms, namely

$$\beta_{-k}|\emptyset\rangle = u_k \wedge |\emptyset_{-1}\rangle + u_0 \wedge u_{k-1} \wedge |\emptyset_{-2}\rangle + \cdots + u_0 \wedge u_{-1} \wedge \cdots \wedge u_{-k+2} \wedge u_1 \wedge |\emptyset_{-k}\rangle.$$

Moreover one can check that each of these k terms is mapped to $|\emptyset\rangle$ by β_k. □

Proposition 1 shows that the endomorphisms β_k $(k \in \mathbb{Z}^*)$ endow $\mathscr{F}$ with an action of an infinite-dimensional Heisenberg Lie algebra $\mathfrak{H}$ with generators $p_i, q_i (i \in \mathbb{N}^*)$, and K, and defining relations

$$[p_i, q_j] = i\delta_{i,-j}K, \quad [K, p_i] = [K, q_i] = 0.$$

In this representation, K acts by $\mathrm{Id}_{\mathbb{F}}$, and $\deg(q_i) = i = -\deg(p_i)$. Its character is given by Eq. (1), so by the classical theory of Heisenberg algebras (see *e.g.* [**K**, §9.13]), we obtain that $\mathscr{F}$ is isomorphic to the irreducible representation of $\mathfrak{H}$ in $\mathscr{B} = \mathbb{C}[x_i \mid i \in \mathbb{N}^*]$ in which $p_i = \partial/\partial x_i$ and q_i is the multiplication by x_i. Note that we endow $\mathscr{B}$ with the unusual grading given by $\deg x_i = i$. Thus we have obtained a canonical isomorphism between the fermionic Fock space $\mathscr{F}$ and the *bosonic Fock space* $\mathscr{B}$. This is called the *boson-fermion correspondence*.

Exercise 5 Identify $\mathscr{B} = \mathbb{C}[x_i \mid i \in \mathbb{N}^*]$ with the ring of symmetric functions by regarding x_i as the degree i power sum (see [**Mcd**]). Then the Schur function s_λ is given by

$$s_\lambda = \sum_\mu \chi_\lambda(\mu) x_\mu / z_\mu,$$

where for λ and $\mu = (1^{k_1}, 2^{k_2}, \ldots)$ partitions of m,

$$x_\mu = x_1^{k_1} x_2^{k_2} \cdots, \qquad z_\mu = 1^{k_1} k_1! 2^{k_2} k_2! \cdots,$$

and $\chi_\lambda(\mu)$ denotes the irreducible character χ_λ of $\mathfrak{S}_m$ evaluated on the conjugacy class of cycle-type μ. Show that the boson-fermion correspondence $\mathscr{F} \to \mathscr{B}$ maps $|\lambda\rangle$ to s_λ [**MJD**, §9.3], [**K**, §14.10].

2.2.8 Decomposition of $\mathscr{F}$

Let us go back to the endomorphisms $b_k^\infty = \beta_{nk}$ $(k \in \mathbb{Z}^*)$ of §2.2.6. By Proposition 1, they generate a Heisenberg subalgebra $\mathfrak{H}_n$ of $\mathrm{End}\mathscr{F}$ with commutation relations

$$[b_k^\infty, b_l^\infty] = nk\, \delta_{k,-l}\, \mathrm{Id}_{\mathscr{F}}, \qquad (k, l \in \mathbb{Z}^*). \tag{9}$$

Since the actions of $\widehat{\mathfrak{g}}$ and $\mathfrak{H}_n$ on $\mathscr{F}$ commute with each other, we can regard $\mathscr{F}$ as a module over the product of enveloping algebras $U(\widehat{\mathfrak{g}}) \otimes U(\mathfrak{H}_n)$. Let $\mathbb{C}[\mathfrak{H}_n^-]$ denote the commutative subalgebra of $U(\mathfrak{H}_n)$ generated by the b_k^∞ $(k < 0)$. The same argument as in §2.2.7 shows that $\mathbb{C}[\mathfrak{H}_n^-]|\emptyset\rangle$ is an irreducible representation of $\mathfrak{H}_n$ with character

$$\prod_{k \geqslant 1} \frac{1}{1 - t^{nk}}. \tag{10}$$

On the other hand, the Chevalley generators of $\widehat{\mathfrak{g}}$ act on $|\emptyset\rangle$ by

$$e_i|\emptyset\rangle = 0, \quad f_i|\emptyset\rangle = \delta_{i,0}|1\rangle, \quad h_i|\emptyset\rangle = \delta_{i,0}|\emptyset\rangle, \qquad (i = 0, 1, \ldots, n-1).$$

It follows from the representation theory of Kac-Moody algebras that $U(\widehat{\mathfrak{g}})|\emptyset\rangle$ is isomorphic to the irreducible $\widehat{\mathfrak{g}}$-module $V(\Lambda_0)$ with level one highest weight Λ_0, whose character is (see *e.g.* [**K**, Ex. 14.3])

$$\prod_{k \geqslant 0} \prod_{i=1}^{n-1} \frac{1}{1 - t^{nk+i}}. \tag{11}$$

By comparing Eq. (1), (10), and (11), we see that we have proved that

Proposition 2 *The $U(\widehat{\mathfrak{g}}) \otimes U(\mathfrak{H}_n)$-modules $\mathscr{F}$ and $V(\Lambda_0) \otimes \mathbb{C}[\mathfrak{H}_n^-]$ are isomorphic.* □

We can now easily solve Exercise 4. Indeed, it follows from Proposition 2 that the subspace of primitive vectors of $\mathscr{F}$ for the action of $\widehat{\mathfrak{g}}$ is $\mathbb{C}[\mathfrak{H}_n^-]$. Thus, for every $k > 0$,

$$b_{-k}^\infty|\emptyset\rangle = \sum_{i=1}^{nk} (-1)^{nk-i} |i, 1^{nk-i}\rangle$$

is a primitive vector. Here, $(i, 1^{nk-i})$ denotes the partition $(i, 1, 1, \ldots, 1)$ with 1 repeated $nk - i$ times.

2.3 The higher level Fock space representations of $\widehat{\mathfrak{sl}}_n$

2.3.1 The representation $\mathscr{F}[\ell]$

Let ℓ be a positive integer. Following Frenkel [**F1**, **F2**], we notice that $\widehat{\mathfrak{g}}$ contains the subalgebra

$$\widehat{\mathfrak{g}}_{[\ell]} := \mathfrak{g} \otimes \mathbb{C}[z^{\ell}, z^{-\ell}] \oplus \mathbb{C}c,$$

and that the linear map $\iota_\ell : \widehat{\mathfrak{g}} \to \widehat{\mathfrak{g}}_{[\ell]}$ given by

$$\iota_\ell(a \otimes z^k) = a \otimes z^{k\ell}, \quad \iota_\ell(c) = \ell c, \qquad (a \in \mathfrak{g},\ k \in \mathbb{Z}),$$

is a Lie algebra isomorphism. Hence by restricting the Fock space representation of $\widehat{\mathfrak{g}}$ to $\widehat{\mathfrak{g}}_{[\ell]}$ we obtain a new Fock space representation $\mathscr{F}[\ell]$ of $\widehat{\mathfrak{g}}$. In this representation, the central element c acts via $\iota_\ell(c) = \ell c$, therefore by multiplication by ℓ. So $\mathscr{F}[\ell]$ has level ℓ and is called *the level ℓ Fock space representation* of $\widehat{\mathfrak{g}}$. More generally, for every $m \in \mathbb{Z}$ we get from the Fock space $\mathscr{F}_m$ of charge m a level ℓ representation $\mathscr{F}_m[\ell]$ of $\widehat{\mathfrak{sl}}_n$.

2.3.2 Antisymmetrizer construction

The representations $\mathscr{F}_m[\ell]$ can also be constructed step by step from the representation $V(z)$ as we did in the case $\ell = 1$. This will help to understand their structure. In particular, it will yield an action of $\widehat{\mathfrak{sl}}_\ell$ of level n on $\mathscr{F}_m[\ell]$, commuting with the level ℓ action of $\widehat{\mathfrak{sl}}_n$.

(a) The new action of $\widehat{\mathfrak{g}}$ on $V(z)$ coming from the identification $\widehat{\mathfrak{g}} \equiv \widehat{\mathfrak{g}}_{[\ell]}$ is as follows:

$$\begin{array}{rclrcll} e_i u_k &=& \delta_{k\equiv i+1}\, u_{k-1}, & f_i u_k &=& \delta_{k\equiv i}\, u_{k+1}, & (1 \leqslant i \leqslant n-1), \\ e_0 u_k &=& \delta_{k\equiv 1}\, u_{k-1-(\ell-1)n}, & f_0 u_k &=& \delta_{k\equiv 0}\, u_{k+1+(\ell-1)n}. & \end{array}$$

Here, $\delta_{k\equiv i}$ is the Kronecker symbol which is equal to 1 if $k \equiv i \bmod n$ and to 0 otherwise. For example, if $n = 2$ and $\ell = 3$, this can be pictured as follows:

$$\begin{array}{ccccccccccccccc} \cdots & \xrightarrow{f_0} & u_{-5} & \xrightarrow{f_1} & u_{-4} & \xrightarrow{f_0} & u_1 & \xrightarrow{f_1} & u_2 & \xrightarrow{f_0} & u_7 & \xrightarrow{f_1} & u_8 & \xrightarrow{f_0} & \cdots \\ \cdots & \xrightarrow{f_0} & u_{-3} & \xrightarrow{f_1} & u_{-2} & \xrightarrow{f_0} & u_3 & \xrightarrow{f_1} & u_4 & \xrightarrow{f_0} & u_9 & \xrightarrow{f_1} & u_{10} & \xrightarrow{f_0} & \cdots \\ \cdots & \xrightarrow{f_0} & u_{-1} & \xrightarrow{f_1} & u_0 & \xrightarrow{f_0} & u_5 & \xrightarrow{f_1} & u_6 & \xrightarrow{f_0} & u_{11} & \xrightarrow{f_1} & u_{12} & \xrightarrow{f_0} & \cdots \end{array}$$

This suggests a different identification of the vector space $V(z)$ with a tensor product. Let us change our notation and write $U := \oplus_i \mathbb{C}u_i$. We consider again

$$V = \bigoplus_{i=1}^{n} \mathbb{C}v_i,$$

and also

$$W = \bigoplus_{j=1}^{\ell} \mathbb{C}w_j.$$

Then we can identify $W \otimes V \otimes \mathbb{C}[z, z^{-1}]$ with U by

$$w_j \otimes v_i \otimes z^k \equiv u_{i+(j-1)n-\ell nk}, \qquad (1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant \ell,\ k \in \mathbb{Z}). \tag{12}$$

Now the previous action of $\widehat{\mathfrak{g}} = \widehat{\mathfrak{sl}}_n$ on the space U reads as follows:

$$\begin{array}{lllllllll} e_i & = & \mathrm{Id}_W \otimes E_i \otimes 1, & f_i & = & \mathrm{Id}_W \otimes F_i \otimes 1, & h_i & = & \mathrm{Id}_W \otimes H_i \otimes 1, \qquad (1 \leqslant i \leqslant n-1), \\ e_0 & = & \mathrm{Id}_W \otimes E_{n1} \otimes z, & f_0 & = & \mathrm{Id}_W \otimes E_{1n} \otimes z^{-1}, & h_0 & = & \mathrm{Id}_W \otimes (E_{nn} - E_{11}) \otimes 1. \end{array}$$

But we also have a commuting action of $\widetilde{\mathfrak{g}} := \widehat{\mathfrak{sl}}_\ell$ given by

$$\begin{array}{lllllllll} \widetilde{e}_i & = & \widetilde{E}_i \otimes \mathrm{Id}_V \otimes 1, & \widetilde{f}_i & = & \widetilde{F}_i \otimes \mathrm{Id}_V \otimes 1, & \widetilde{h}_i & = & \widetilde{H}_i \otimes \mathrm{Id}_V \otimes 1, \qquad (1 \leqslant i \leqslant \ell-1), \\ \widetilde{e}_0 & = & \widetilde{E}_{\ell 1} \otimes \mathrm{Id}_V \otimes z, & \widetilde{f}_0 & = & \widetilde{E}_{1\ell} \otimes \mathrm{Id}_V \otimes z^{-1}, & \widetilde{h}_0 & = & (\widetilde{E}_{\ell\ell} - \widetilde{E}_{11}) \otimes \mathrm{Id}_V \otimes 1. \end{array}$$

Here, we have denoted by $\widetilde{E}_{ij}$ the matrix units of $\mathfrak{gl}_\ell$, and we have set $\widetilde{E}_i = \widetilde{E}_{i,i+1}$, $\widetilde{F}_i = \widetilde{E}_{i+1,i}$ and $\widetilde{H}_i = \widetilde{E}_{i,i} - \widetilde{E}_{i+1,i+1}$.

(b) From this we get as in §2.1.4 some tensor representations. Let $r \in \mathbb{N}^*$. We have

$$U^{\otimes r} = (W \otimes V \otimes \mathbb{C}[z, z^{-1}])^{\otimes r} \cong W^{\otimes r} \otimes V^{\otimes r} \otimes \mathbb{C}[z_1^{\pm}, \ldots, z_r^{\pm}].$$

This inherits from U two commuting actions of $\widehat{\mathfrak{g}}$ and of $\widetilde{\mathfrak{g}}$.

We have endowed $V^{\otimes r} \otimes \mathbb{C}[z_1^{\pm}, \ldots, z_r^{\pm}]$ with a left action of $\widehat{\mathfrak{S}}_r$ in §2.1.5. Similarly, we endow $W^{\otimes r}$ with a right action of $\mathfrak{S}_r$, given by

$$w_{i_1} \otimes \cdots \otimes w_{i_r} \cdot \sigma = \varepsilon(\sigma)(w_{i_{\sigma(1)}} \otimes \cdots \otimes w_{i_{\sigma(r)}}) \qquad (\sigma \in \mathfrak{S}_r),$$

where $\varepsilon(\sigma)$ denotes the sign of the permutation σ.

(c) We can now pass to the wedge product and consider the representation $\wedge^r U$, on which $\widehat{\mathfrak{g}}$ and $\widetilde{\mathfrak{g}}$ act naturally. It has a standard basis consisting of normally ordered wedges

$$\wedge u_{\mathbf{i}} := u_{i_1} \wedge \cdots \wedge u_{i_r}, \qquad (i_1 > \cdots > i_r \in \mathbb{Z}).$$

Proposition 3 *We have the following isomorphism of vector spaces*

$$\wedge^r U \cong W^{\otimes r} \otimes_{\mathbb{C}\mathfrak{S}_r} \left(V^{\otimes r} \otimes \mathbb{C}[z_1^{\pm}, \ldots, z_r^{\pm}] \right).$$

Proof — Let $\mathbf{1}^-$ denote the one-dimensional sign representation of $\mathfrak{S}_r$. We have

$$\wedge^r U \cong \mathbf{1}^- \otimes_{\mathbb{C}\mathfrak{S}_r} U^{\otimes r},$$

where $\mathfrak{S}_r$ acts on $U^{\otimes r}$ by permuting the factors. Let A and B be two vector spaces. Then it is easy to see that

$$\mathbf{1}^- \otimes_{\mathbb{C}\mathfrak{S}_r} (A \otimes B)^{\otimes r} \cong A^{\otimes r} \otimes_{\mathbb{C}\mathfrak{S}_r} B^{\otimes r},$$

where $\mathfrak{S}_r$ acts on $B^{\otimes r}$ by permuting the factors, and on $A^{\otimes r}$ by permuting the factors and multiplying by the sign of the permutation. Recall that the left action of $\mathfrak{S}_r$ on $(V \otimes \mathbb{C}[z^{\pm}])^{\otimes r}$ is by permutation of the factors, while the right action of $\mathfrak{S}_r$ on $W^{\otimes r}$ is by signed permutation of the factors. It follows that

$$\wedge^r U \cong W^{\otimes r} \otimes_{\mathbb{C}\mathfrak{S}_r} \left(V \otimes \mathbb{C}[z^{\pm}] \right)^{\otimes r} \cong W^{\otimes r} \otimes_{\mathbb{C}\mathfrak{S}_r} \left(V^{\otimes r} \otimes \mathbb{C}[z_1^{\pm}, \ldots, z_r^{\pm}] \right).$$

□

(d) The elements $b_k = \sum_{i=1}^r z_i^k \in \mathbb{C}\widehat{\mathfrak{S}}_r$ commute with $\mathfrak{S}_r \subset \widehat{\mathfrak{S}}_r$, hence their action on

$$U^{\otimes r} \cong W^{\otimes r} \otimes V^{\otimes r} \otimes \mathbb{C}[z_1^{\pm}, \ldots, z_r^{\pm}]$$

by multiplication on the last factor descends to $\wedge^r U$. Using Eq. (12), we see that it is given on the basis of normally ordered wedges by

$$b_k(\wedge u_{\mathbf{i}}) = u_{i_1 - n\ell k} \wedge u_{i_2} \wedge \cdots \wedge u_{i_r} + u_{i_1} \wedge u_{i_2 - n\ell k} \wedge \cdots \wedge u_{i_r} + \cdots + u_{i_1} \wedge u_{i_2} \wedge \cdots \wedge u_{i_r - n\ell k}. \quad (13)$$

This action commutes with the actions of $\widehat{\mathfrak{g}}$ and $\widetilde{\mathfrak{g}}$.

(e) Finally, we can pass to the limit $r \to \infty$ as in §2.2. For every charge $m \in \mathbb{Z}$, we obtain exactly in the same way a Fock space with a standard basis consisting of all infinite wedges

$$\wedge u_{\mathbf{i}} := u_{i_1} \wedge u_{i_2} \wedge \cdots \wedge u_{i_r} \wedge \cdots, \qquad (\mathbf{i} = (i_1 > i_2 > \cdots > i_r > \cdots) \in \mathbb{Z}^{\mathbb{N}^*}),$$

which coincide, except for finitely many indices, with

$$|\emptyset_m\rangle := u_m \wedge u_{m-1} \wedge \cdots \wedge u_{m-r} \wedge u_{m-r-1} \wedge \cdots$$

We shall denote that space by $\mathscr{F}_m[\ell]$, since it is equipped with a level ℓ action of $\widehat{\mathfrak{g}}$, obtained, as in §2.2.4, §2.2.5, by passing to the limit $r \to \infty$ in the action of **(c)**. It is also equipped with commuting actions of $\widetilde{\mathfrak{g}}$ of level n, and of the Heisenberg algebra $\mathfrak{H}_{n\ell}$ generated by the limits b_k^∞ of the operators b_k of Eq. (13), which satisfy

$$[b_j^\infty, b_k^\infty] = n\ell j\, \delta_{j,-k} \mathrm{Id}_{\mathscr{F}_m[\ell]}, \qquad (j,k \in \mathbb{Z}^*). \quad (14)$$

2.3.3 Labellings of the standard basis

It is convenient to label the standard basis of $\mathscr{F}_m[\ell]$ by partitions. In fact, we shall use two different labellings. The first one is as in §2.2.1, namely, for $\wedge u_{\mathbf{i}} \in \mathscr{F}_m[\ell]$ we set

$$\lambda_k = i_k - m - k + 1, \qquad (k \in \mathbb{N}^*).$$

By the definition of $\mathscr{F}_m[\ell]$, the weakly decreasing sequence $\lambda = (\lambda_k \mid k \in \mathbb{N}^*)$ is zero for k large enough, hence is a partition. The first labelling is

$$\wedge u_{\mathbf{i}} = |\lambda, m\rangle. \quad (15)$$

For the second labelling, we use Eq. (12) and write, for every $k \in \mathbb{N}^*$,

$$u_{i_k} = w_{b_k} \otimes v_{a_k} \otimes z^{c_k}, \qquad (1 \leqslant a_k \leqslant n,\ 1 \leqslant b_k \leqslant \ell,\ c_k \in \mathbb{Z}).$$

This defines sequences (a_k), (b_k), and (c_k). For each $t = 1, \ldots, \ell$ consider the infinite sequence $k_1^{(t)} < k_2^{(t)} < \cdots$, consisting of all integers k such that $b_k = t$. Then it is easy to check that the sequence

$$a_{k_1^{(t)}} - nc_{k_1^{(t)}}, \quad a_{k_2^{(t)}} - nc_{k_2^{(t)}}, \quad \ldots \quad (16)$$

is strictly decreasing and contains all negative integers except possibly a finite number of them. Therefore there exists a unique integer m_t such that the sequence

$$\lambda_1^{(t)} = a_{k_1^{(t)}} - nc_{k_1^{(t)}} - m_t, \quad \lambda_2^{(t)} = a_{k_2^{(t)}} - nc_{k_2^{(t)}} - m_t - 1, \quad \lambda_3^{(t)} = a_{k_3^{(t)}} - nc_{k_3^{(t)}} - m_t - 2, \quad \ldots$$

is a partition $\lambda^{(t)}$. Let $\underline{\lambda}_\ell = (\lambda^{(1)}, \ldots, \lambda^{(\ell)})$ be the ℓ-tuple of partitions thus obtained, and let $\mathbf{m}_\ell = (m_1, \ldots, m_\ell)$. The second labelling is

$$\wedge u_{\mathbf{i}} = |\underline{\lambda}_\ell, \mathbf{m}_\ell\rangle. \quad (17)$$

Example 1 Let $n = 2$ and $\ell = 3$. Consider $\wedge u_{\mathbf{i}}$ given by the following sequence

$$\mathbf{i} = (7,\ 6,\ 4,\ 3,\ 1, -1, -2, -3, -5, -6, \ldots)$$

containing all integers $\leqslant -5$. It differs only in finitely many places from the sequence

$$(3,\ 2,\ 1,\ 0, -1, -2, -3, -4, -5, -6, \ldots).$$

Hence $m = 3$ and $\lambda = (4,4,3,3,2,1,1,1)$.

For the second labelling, we have (use the picture in §2.3.2 (a))

$$\begin{array}{lllllllll}
u_7 & = & w_1 \otimes v_1 \otimes z^{-1}, & u_6 & = & w_3 \otimes v_2 \otimes z^0, & u_4 & = & w_2 \otimes v_2 \otimes z^0, \\
u_3 & = & w_2 \otimes v_1 \otimes z^0, & u_1 & = & w_1 \otimes v_1 \otimes z^0, & u_{-1} & = & w_3 \otimes v_1 \otimes z^1, \\
u_{-2} & = & w_2 \otimes v_2 \otimes z^1, & u_{-3} & = & w_2 \otimes v_1 \otimes z^1, & u_{-5} & = & w_1 \otimes v_1 \otimes z^1, \\
& \cdots & & & \cdots & & & \cdots &
\end{array}$$

Hence the sequences in Eq. (16) are

$$\begin{array}{ll}
3,\ 1,\ -1,\ -2,\ -3,\ \ldots & (t = 1), \\
2,\ 1,\ 0,\ -1,\ -2,\ \ldots & (t = 2), \\
2,\ -1,\ -2,\ -3, \ldots & (t = 3).
\end{array}$$

It follows that $\mathbf{m}_3 = (1,2,0)$ and $\underline{\lambda}_3 = ((2,1), \emptyset, (2))$.

Exercise 6 Prove that $m_1 + \cdots + m_\ell = m$. Show that the map $(\lambda, m) \mapsto (\underline{\lambda}_\ell, \mathbf{m}_\ell)$ is a bijection from $\mathscr{P} \times \mathbb{Z}$ to $\mathscr{P}^\ell \times \mathbb{Z}^\ell$.

2.3.4 The spaces $\mathbf{F}[\mathbf{m}_\ell]$

Define $\mathbb{Z}^\ell(m) = \{(m_1, \ldots, m_\ell) \in \mathbb{Z}^\ell \mid m_1 + \cdots + m_\ell = m\}$. Given $\mathbf{m}_\ell \in \mathbb{Z}^\ell(m)$, let $\mathbf{F}[\mathbf{m}_\ell]$ be the subspace of $\mathscr{F}_m[\ell]$ spanned by all vectors of the standard basis of the form $|\underline{\lambda}_\ell, \mathbf{m}_\ell\rangle$ for some ℓ-tuple of partitions $\underline{\lambda}_\ell$. Using Exercise 6, we get the decomposition of vector spaces

$$\mathscr{F}_m[\ell] = \bigoplus_{\mathbf{m}_\ell \in \mathbb{Z}^\ell(m)} \mathbf{F}[\mathbf{m}_\ell]. \tag{18}$$

Moreover, since the action of $\widehat{\mathfrak{g}}$ on $W \otimes V \otimes \mathbb{C}[z^\pm]$ involves only the last two factors and leaves W untouched, it is easy to see that every summand $\mathbf{F}[\mathbf{m}_\ell]$ is stable under $\widehat{\mathfrak{g}}$. Similarly, since the action of the bosons b_k involves only the factor $\mathbb{C}[z_1^\pm, \ldots, z_r^\pm]$ in $U^{\otimes r} = W^{\otimes r} \otimes V^{\otimes r} \otimes \mathbb{C}[z_1^\pm, \ldots, z_r^\pm]$, each space $\mathbf{F}[\mathbf{m}_\ell]$ is stable under the Heisenberg algebra $\mathfrak{H}_{n\ell}$. Therefore Eq. (18) is in fact a decomposition of $U(\widehat{\mathfrak{g}}) \otimes U(\mathfrak{H}_{n\ell})$-modules.

The space $\mathbf{F}[\mathbf{m}_\ell]$ is called the *level ℓ Fock space with multi-charge* $\mathbf{m}_\ell$.

Remark 1 When $\ell = 1$, $\mathbf{F}[\mathbf{m}_1] = \mathscr{F}_{m_1}$ is irreducible as a $U(\widehat{\mathfrak{g}}) \otimes U(\mathfrak{H}_n)$-module (see Proposition 2). But for $\ell > 1$, the spaces $\mathbf{F}[\mathbf{m}_\ell]$ are in general *not* irreducible as $U(\widehat{\mathfrak{g}}) \otimes U(\mathfrak{H}_{n\ell})$-modules.

2.3.5 ℓ-tuples of Young diagrams

Let us generalize the combinatorial definitions of §2.2.2 to ℓ-tuples of Young diagrams. Consider an ℓ-tuple of partitions $\underline{\lambda}_\ell = (\lambda^{(1)}, \ldots, \lambda^{(\ell)})$. We may represent it as an ℓ-tuple of Young diagrams. Given $\mathbf{m}_\ell = (m_1, \ldots, m_\ell) \in \mathbb{Z}^\ell$, we attach to each cell γ of these diagrams a *content* $c(\gamma)$. If γ is the cell of the diagram $\lambda^{(d)}$ in column number i and row number j then

$$c(\gamma) = m_d + i - j. \tag{19}$$

For example let $\underline{\lambda}_\ell = ((2,1),(2,2),(3,2,1))$ and $\mathbf{m}_\ell = (2,0,3)$. The contents of the cells of $(\mathbf{m}_\ell, \underline{\lambda}_\ell)$ are

$$\begin{array}{|c|c}\hline 1 & \\ \hline 2 & \multicolumn{1}{c|}{3} \\ \hline \end{array}\,, \qquad \begin{array}{|c|c|}\hline -1 & 0 \\ \hline 0 & 1 \\ \hline \end{array}\,, \qquad \begin{array}{|c|c|c|}\hline 1 & \multicolumn{2}{c}{} \\ \cline{1-2} 2 & 3 & \\ \hline 3 & 4 & 5 \\ \hline \end{array}\,.$$

The pair $(\mathbf{m}_\ell, \underline{\lambda}_\ell)$ is called a *charged ℓ-tuple of Young diagrams*. If $i \in \{0,1,\ldots,n-1\}$, we say that a cell γ of $(\mathbf{m}_\ell, \underline{\lambda}_\ell)$ is an *i-cell* if its content $c(\gamma)$ is congruent to i modulo n.

2.3.6 Action of the Chevalley generators on $\mathbf{F}[\mathbf{m}_\ell]$

By unwinding the definition of the action of $\widehat{\mathfrak{g}}$ on $\mathbf{F}[\mathbf{m}_\ell]$, and the correspondence between the labellings of the standard basis, one obtains the following simple combinatorial formulas for the action of the Chevalley generators, which generalize those of §2.2.4 and §2.2.5.

Proposition 4 *We have*

$$f_i|\underline{\lambda}_\ell, \mathbf{m}_\ell\rangle = \sum |\underline{\mu}_\ell, \mathbf{m}_\ell\rangle, \tag{20}$$

where the sum is over all $\underline{\mu}_\ell$ obtained from $\underline{\lambda}_\ell$ by adding an i-cell. Similarly,

$$e_i|\underline{\mu}_\ell, \mathbf{m}_\ell\rangle = \sum |\underline{\lambda}_\ell, \mathbf{m}_\ell\rangle, \tag{21}$$

where the sum is over all $\underline{\lambda}_\ell$ obtained from $\underline{\mu}_\ell$ by removing an i-cell. □

Exercise 7 Take $n = 2$, $\ell = 3$, $\mathbf{m}_3 = (0,0,1)$, and $\underline{\lambda}_3 = ((1), (2), (1,1))$. Check that

$$\begin{aligned}
e_0|\underline{\lambda}_3, \mathbf{m}_3\rangle &= (\emptyset,(2),(1,1)) + ((1),(2),(1)),\\
e_1|\underline{\lambda}_3, \mathbf{m}_3\rangle &= ((1),(1),(1,1)),\\
f_0|\underline{\lambda}_3, \mathbf{m}_3\rangle &= ((1),(3),(1,1)) + ((1),(2),(2,1)),\\
f_1|\underline{\lambda}_3, \mathbf{m}_3\rangle &= ((2),(2),(1,1)) + ((1,1),(2),(1,1)) + ((1),(2,1),(1,1)) + ((1),(2),(1,1,1)).
\end{aligned}$$

Denote by r_k $(1 \leqslant k \leqslant \ell)$ the residue of m_k modulo n. It follows from Eq. (20), (21) that $|\emptyset, \mathbf{m}_\ell\rangle$ is a primitive vector of $\mathbf{F}[\mathbf{m}_\ell]$, of weight

$$\Lambda_{\mathbf{m}_\ell} = \sum_{k=1}^{\ell} \Lambda_{r_k}.$$

Exercise 8 Check it.

Hence we see that the submodule $U(\widehat{\mathfrak{g}})|\emptyset, \mathbf{m}_\ell\rangle$ is isomorphic to the irreducible module $V(\Lambda_{\mathbf{m}_\ell})$. In particular, every integrable highest weight irreducible $\widehat{\mathfrak{g}}$-module can be realized as a submodule of a Fock space representation.

The formulas (20), (21) also show that the action of the Chevalley generators, and hence the isomorphism type of $\mathbf{F}[\mathbf{m}_\ell]$, only depends on the residues modulo n of the components of the multi-charge $\mathbf{m}_\ell$. Therefore we do not lose anything by assuming that $\mathbf{m}_\ell \in \{0,1,\ldots,n-1\}^\ell$.

This is in sharp contrast with the quantum case, where it will be important to deal with multi-charges $\mathbf{m}_\ell \in \mathbb{Z}^\ell$.

3 Fock space representation of $U_q(\widehat{\mathfrak{sl}}_n)$: level 1

In this section we study a q-analogue of the constructions of §2.1 and §2.2.

3.1 The quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_n)$

The enveloping algebra $U(\widehat{\mathfrak{g}})$ has a q-analogue introduced by Drinfeld and Jimbo. This is the algebra $U_q(\widehat{\mathfrak{g}})$ over $\mathbb{Q}(q)$ with generators e_i, f_i, t_i, t_i^{-1} $(0 \leqslant i \leqslant n-1)$ subject to the following relations:

$$t_i t_i^{-1} = t_i^{-1} t_i = 1, \qquad t_i t_j = t_j t_i$$

$$t_i e_j t_i^{-1} = q^{a_{ij}} e_j, \qquad t_i f_j t_i^{-1} = q^{-a_{ij}} f_j, \qquad e_i f_j - f_j e_i = \delta_{ij} \frac{t_i - t_i^{-1}}{q - q^{-1}},$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_q e_i^{1-a_{ij}-k} e_j e_i^k = 0 \quad (i \neq j),$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_q f_i^{1-a_{ij}-k} f_j f_i^k = 0 \quad (i \neq j).$$

Here, $A = [a_{ij}]_{0 \leqslant i,j \leqslant n-1}$ is the Cartan matrix of type $A_{n-1}^{(1)}$, and

$$\begin{bmatrix} m \\ k \end{bmatrix}_q = \frac{[m][m-1]\cdots[m-k+1]}{[k][k-1]\cdots[1]}$$

is the q-analogue of a binomial coefficient, where $[k] = 1 + q + \cdots + q^{k-1}$.

Exercise 9 Check that $K := t_0 t_1 \cdots t_{n-1}$ is central in $U_q(\widehat{\mathfrak{g}})$.

The q-analogue of the "vector" representation $V(z)$ of $\widehat{\mathfrak{g}}$ (see §2.1.3) is the $\mathbb{Q}(q)$-vector space

$$U := \bigoplus_{k \in \mathbb{Z}} \mathbb{Q}(q) u_k,$$

with the $U_q(\widehat{\mathfrak{g}})$-action given by

$$e_i u_k = \delta_{k \equiv i+1} u_{k-1}, \quad f_i u_k = \delta_{k \equiv i} u_{k+1}, \quad t_i u_k = q^{\delta_{k \equiv i} - \delta_{k \equiv i+1}} u_k.$$

Exercise 10 Check that this defines a representation of $U_q(\widehat{\mathfrak{g}})$. Check that K acts on U by multiplication by $q^0 = 1$, *i.e.* U is a level 0 representation of $U_q(\widehat{\mathfrak{g}})$.

3.2 The tensor representations

The algebra $U_q(\widehat{\mathfrak{g}})$ is a Hopf algebra with comultiplication Δ given by

$$\Delta f_i = f_i \otimes 1 + t_i \otimes f_i, \quad \Delta e_i = e_i \otimes t_i^{-1} + 1 \otimes e_i, \quad \Delta t_i^{\pm} = t_i^{\pm} \otimes t_i^{\pm}. \tag{22}$$

This allows to endow the tensor powers $U^{\otimes r}$ with the structure of a $U_q(\widehat{\mathfrak{g}})$-module. Let us write $u_{\mathbf{i}} := u_{i_1} \otimes \cdots \otimes u_{i_r}$ for $\mathbf{i} = (i_1, \ldots, i_r) \in \mathbb{Z}^r$. We then have

$$f_k u_{\mathbf{i}} = \sum_{\substack{j=1 \\ i_j \equiv k}}^{r} q^{\sum_{l=1}^{j-1} (\delta_{i_l \equiv k} - \delta_{i_l \equiv k+1})} u_{\mathbf{i} + \varepsilon_j}, \tag{23}$$

$$e_k u_{\mathbf{i}} = \sum_{\substack{j=1 \\ i_j \equiv k+1}}^{r} q^{-\sum_{l=j+1}^{r} (\delta_{i_l \equiv k} - \delta_{i_l \equiv k+1})} u_{\mathbf{i} - \varepsilon_j}, \tag{24}$$

where $(\varepsilon_j \mid 1 \leqslant j \leqslant r)$ is the canonical basis of $\mathbb{Z}^r$.

The q-analogue of the action of $\widehat{\mathfrak{S}}_r$ on $V(z)^{\otimes r}$ given in §2.1.5 is an action of the affine Hecke algebra.

3.3 The affine Hecke algebra $\widehat{H}_r$

$\widehat{H}_r$ is the algebra over $\mathbb{Q}(q)$ generated by T_i $(1 \leqslant i \leqslant r-1)$ and invertible elements Y_i $(1 \leqslant i \leqslant r)$ subject to the relations

$$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, \tag{25}$$
$$T_iT_j = T_jT_i, \qquad (i-j \neq \pm 1), \tag{26}$$
$$(T_i - q^{-1})(T_i + q) = 0, \tag{27}$$
$$Y_iY_j = Y_jY_i, \tag{28}$$
$$T_iY_j = Y_jT_i \text{ for } j \neq i, i+1, \tag{29}$$
$$T_iY_iT_i = Y_{i+1}. \tag{30}$$

The generators T_i replace the simple transpositions s_i of $\mathfrak{S}_r$, and the generators Y_i replace the generators z_i of $\widehat{\mathfrak{S}}_r$. Because of Eq. (25) (26), for every $w \in \mathfrak{S}_r$ written in reduced form as $w = s_{i_1} \cdots s_{i_k}$ we can define $T_w := T_{i_1} \cdots T_{i_k}$. This does not depend on the choice of a reduced expression. For $\mathbf{i} = (i_1, \ldots, i_r) \in \mathbb{Z}^r$ we shall also write $Y^{\mathbf{i}} := Y_1^{i_1} \cdots Y_r^{i_r}$.

There is a canonical involution $x \mapsto \overline{x}$ of $\widehat{H}_r$ defined as the unique $\mathbb{Q}$-algebra automorphism such that

$$\overline{q} = q^{-1}, \quad \overline{T_i} = T_i^{-1}, \quad \overline{Y_i} = T_{w_0}^{-1} Y_{r-i+1} T_{w_0},$$

where w_0 is the longest element of $\mathfrak{S}_r$. We have the following more general formula, which can be deduced from the commutation formulas.

Proposition 5 *For $s \in \mathfrak{S}_r$ and $\mathbf{i} \in \mathbb{Z}^r$, we have:* $\overline{(Y^{\mathbf{i}}T_s)} = T_{w_0}^{-1} Y^{w_0 \cdot \mathbf{i}} T_{w_0 s}$. □

3.4 Action of $\widehat{H}_r$ on $U^{\otimes r}$

Recall the fundamental domain $A_r := \{\mathbf{i} \in \mathbb{Z}^r \mid 1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_r \leqslant n\}$ of §2.1.5 for the action of $\widehat{\mathfrak{S}}_r$. We write $\mathfrak{S}_r A_r := \{\mathbf{j} \in \mathbb{Z}^r \mid 1 \leqslant j_1, j_2, \cdots, j_r \leqslant n\}$, and we introduce an action of $\widehat{H}_r$ on $U^{\otimes r}$ by requiring that

$$Y_j u_{\mathbf{i}} \;=\; u_{z_j \mathbf{i}}, \qquad (1 \leqslant j \leqslant r,\ \mathbf{i} \in \mathbb{Z}^r), \tag{31}$$

$$T_k u_{\mathbf{j}} \;=\; \begin{cases} u_{s_k \mathbf{j}} & \text{if } j_k < j_{k+1}, \\ q^{-1} u_{\mathbf{j}} & \text{if } j_k = j_{k+1}, \\ u_{s_k \mathbf{j}} + (q^{-1} - q) u_{\mathbf{j}} & \text{if } j_k > j_{k+1}, \end{cases} \qquad (1 \leqslant k \leqslant r-1,\ \mathbf{j} \in \mathfrak{S}_r A_r). \tag{32}$$

In these formulas, $z_j \mathbf{i}$ and $s_k \mathbf{j}$ are as defined in §2.1.5.

Proposition 6 *Eq. (31) (32) define an action of $\widehat{H}_r$ on $U^{\otimes r}$ which commutes with the action of $U_q(\widehat{\mathfrak{g}})$.*

Proof — First we explain how to calculate $T_k u_{\mathbf{i}}$ for an arbitrary $\mathbf{i} \in \mathbb{Z}^r$. Write $\mathbf{i} = \mathbf{j} - n\mathbf{l}$, with $\mathbf{j} \in \mathfrak{S}_r A_r$ and $\mathbf{l} \in \mathbb{Z}^r$ (this is always possible, in a unique way). Then by the first formula $u_{\mathbf{i}} = Y^{\mathbf{l}} u_{\mathbf{j}}$. Write

$$Y^{\mathbf{l}} = Y_k^{l_k - l_{k+1}} \left(Y_1^{l_1} \cdots Y_{k-1}^{l_{k-1}} (Y_k Y_{k+1})^{l_{k+1}} Y_{k+2}^{l_{k+2}} \cdots Y_r^{l_r} \right).$$

Eq. (29) (30) show that T_k commutes with the second bracketed factor. Indeed, one has for example

$$T_k Y_k Y_{k+1} = Y_{k+1} T_k^{-1} Y_{k+1} = Y_{k+1} Y_k T_k.$$

Hence we are reduced to calculate $T_k Y_k^{l_k - l_{k+1}} u_{\mathbf{j}}$. For this, we use the following commutation relations, which are immediate consequences of Eq. (29) (30):

$$T_k Y_k^l = \begin{cases} Y_{k+1}^l T_k + (q - q^{-1}) \displaystyle\sum_{j=1}^{l} Y_k^{l-j} Y_{k+1}^j, & (l \geqslant 0), \\ Y_{k+1}^l T_k + (q^{-1} - q) \displaystyle\sum_{j=1}^{-l} Y_k^{-j} Y_{k+1}^{j+l}, & (l < 0). \end{cases} \tag{33}$$

These relations allow us to express $T_k u_{\mathbf{i}}$ as a linear combination of terms of the form $Y^{\mathbf{m}} T_k u_{\mathbf{j}}$ for some $\mathbf{m} \in \mathbb{Z}^r$ and therefore to calculate $T_k u_{\mathbf{i}}$.

Eq. (32) is copied from some classical formulas of Jimbo [**J1**] which give an action of the finite Hecke algebra $H_r = \langle T_k \mid 1 \leqslant k \leqslant r-1 \rangle$ on the tensor power $V^{\otimes r}$ of the vector representation of $U_q(\mathfrak{sl}_n)$. This action commutes with the action of $U_q(\mathfrak{sl}_n)$ (quantum Schur-Weyl duality). The above discussion shows that Eq. (31) allows to extend this action to an action of $\widehat{H}_r$ on $U^{\otimes r}$. Comparing Eq. (31) and Eq. (23) (24), we see easily that the action of $U_q(\widehat{\mathfrak{g}})$ commutes with the action of the Y_j's. The fact that the action of T_k commutes with e_0 and f_0 can be checked by a direct calculation. □

For $\mathbf{i} \in A_r$, let $H_{\mathbf{i}}$ be the subalgebra of $\widehat{H}_r$ generated by the the T_k's such that $s_k \mathbf{i} = \mathbf{i}$. This is the parabolic subalgebra attached to the parabolic subgroup $\mathfrak{S}_{\mathbf{i}}$. Denote by $\mathbf{1}_{\mathbf{i}}^+$ the one-dimensional $H_{\mathbf{i}}$-module on which T_k acts by multiplication by q^{-1}. Then, Eq. (31) (32) show that the $\widehat{H}_r$-module $U^{\otimes r}$ decomposes as

$$U^{\otimes r} = \bigoplus_{\mathbf{i} \in A_r} \widehat{H}_r u_{\mathbf{i}},$$

and that

$$\widehat{H}_r u_{\mathbf{i}} = \bigoplus_{\sigma \in \widehat{\mathfrak{S}}_r / \mathfrak{S}_{\mathbf{i}}} \mathbb{Q}(q) u_{\sigma \mathbf{i}} \cong \widehat{H}_r \otimes_{H_{\mathbf{i}}} \mathbf{1}_{\mathbf{i}}^+.$$

Hence, $U^{\otimes r}$ is a direct sum of a finite number of *parabolic* $\widehat{H}_r$-modules, parametrized by $\mathbf{i} \in A_r$. Each of these submodules inherits a bar involution, namely, the semi-linear map given by

$$\overline{q} = q^{-1}, \qquad \overline{x u_{\mathbf{i}}} = \overline{x} u_{\mathbf{i}}, \quad (x \in \widehat{H}_r).$$

The following formula follows easily from Proposition 5.

Proposition 7 *Let* $\mathbf{i} \in A_r$ *and* $\mathbf{j} \in \widehat{\mathfrak{S}}_r \mathbf{i}$. *Then*

$$\overline{u_{\mathbf{j}}} = q^{-l(w_{0,\mathbf{i}})} T_{w_0}^{-1} u_{w_0 \mathbf{j}},$$

where $w_{0,\mathbf{i}}$ *is the longest element of* $\mathfrak{S}_{\mathbf{i}}$. □

3.5 The q-deformed wedge-spaces

Let $\mathscr{I}_r := \sum_{k=1}^{r-1} \mathrm{Im}\,(T_k + q\mathrm{Id}) \subset U^{\otimes r}$ be the q-analogue of the subspace of "partially symmetric tensors" (see §2.1.6). We define

$$\wedge_q^r U := U^{\otimes r} / \mathscr{I}_r.$$

Let $\mathrm{pr} : U^{\otimes r} \to \wedge_q^r U$ be the natural projection. For $\mathbf{i} \in \mathbb{Z}^r$, we put

$$\wedge_q u_{\mathbf{i}} := q^{-l(w_0)} \mathrm{pr}\,(u_{\mathbf{i}}).$$

We shall also write $\wedge_q u_{\mathbf{i}} = u_{i_1} \wedge_q u_{i_2} \wedge_q \cdots \wedge_q u_{i_r}$.

The next proposition gives a set of straightening rules to express any $\wedge_q u_{\mathbf{i}}$ in terms of $\wedge_q u_{\mathbf{k}}$'s with $k_1 > \cdots > k_r$.

Proposition 8 *Let $\mathbf{i} \in \mathbb{Z}^r$ be such that $i_k < i_{k+1}$. Write $i_{k+1} = i_k + an + b$, with $a \geqslant 0$ and $0 \leqslant b < n$. Then*

$$\begin{aligned}
\wedge_q u_{\mathbf{i}} &= -u_{i_1} \wedge_q \cdots \wedge_q u_{i_{k+1}} \wedge_q u_{i_k} \wedge_q \cdots \wedge_q u_{i_r}, && \textit{if } b = 0, \\
\wedge_q u_{\mathbf{i}} &= -q^{-1} u_{i_1} \wedge_q \cdots \wedge_q u_{i_{k+1}} \wedge_q u_{i_k} \wedge_q \cdots \wedge_q u_{i_r}, && \textit{if } a = 0, \\
\wedge_q u_{\mathbf{i}} &= -q^{-1} u_{i_1} \wedge_q \cdots \wedge_q u_{i_{k+1}} \wedge_q u_{i_k} \wedge_q \cdots \wedge_q u_{i_r} \\
&\quad -u_{i_1} \wedge_q \cdots \wedge_q u_{i_k+an} \wedge_q u_{i_{k+1}-an} \wedge_q \cdots \wedge_q u_{i_r} \\
&\quad -q^{-1} u_{i_1} \wedge_q \cdots \wedge_q u_{i_{k+1}-an} \wedge_q u_{i_k+an} \wedge_q \cdots \wedge_q u_{i_r}, && \textit{otherwise.}
\end{aligned}$$

Proof — To simplify the notation, let us write $l = i_k$ and $m = i_{k+1}$. Since the relations only involve components k and $k+1$ we shall also use the shorthand notations

$$\begin{aligned}
(j,p) &:= u_{i_1} \otimes \cdots \otimes u_{i_{k-1}} \otimes u_j \otimes u_p \otimes u_{i_{k+2}} \otimes \cdots \otimes u_{i_r} \in U^{\otimes r}, \\
|j,p\rangle &:= u_{i_1} \wedge_q \cdots \wedge_q u_{i_{k-1}} \wedge_q u_j \wedge_q u_p \wedge_q u_{i_{k+2}} \wedge_q \cdots \wedge_q u_{i_r} \in \wedge_q{}^r U.
\end{aligned}$$

We shall use the easily checked fact that $Y_k^p + Y_{k+1}^p$ commutes with T_k for every $p \in \mathbb{Z}$.

Suppose $b = 0$. It follows from Eq. (32) that $T_k(l,l) = q^{-1}(l,l)$. Hence $(l,l) \in \mathrm{Im}\,(T_k + q)$. Since $(Y_k^{-a} + Y_{k+1}^{-a})(T_k + q) = (T_k + q)(Y_k^{-a} + Y_{k+1}^{-a})$ we also have

$$(Y_k^{-a} + Y_{k+1}^{-a})(l,l) = (m,l) + (l,m) \in \mathrm{Im}\,(T_k + q),$$

and thus $|l,m\rangle + |m,l\rangle = 0$.

Suppose $a = 0$. Then $T_k(l,m) = (m,l)$ by Eq. (32), and

$$(T_k + q)(l,m) = (m,l) + q(l,m) \in \mathrm{Im}\,(T_k + q),$$

which gives $|l,m\rangle = -q^{-1}|m,l\rangle$.

Finally suppose that $a, b > 0$. By the previous case $(m, l+an) + q(l+an, m) \in \mathrm{Im}\,(T_k + q)$. Applying $Y_k^a + Y_{k+1}^a$ we get that

$$(m,l) + (m-an, l+an) + q(l,m) + q(l+an, m-an) \in \mathrm{Im}\,(T_k + q),$$

which gives the third claim. □

Exercise 11 Take $r = 2$ and $n = 2$. Check that

$$u_1 \wedge_q u_4 = -q^{-1} u_4 \wedge_q u_1 - u_3 \wedge_q u_2 - q^{-1} u_2 \wedge_q u_3,$$

and that $u_2 \wedge_q u_3 = -q^{-1} u_3 \wedge_q u_2$. Thus

$$u_1 \wedge_q u_4 = -q^{-1} u_4 \wedge_q u_1 + (q^{-2} - 1)\, u_3 \wedge_q u_2.$$

Proposition 9 *Let $\mathbb{Z}^r_> = \{\mathbf{i} \in \mathbb{Z}^r \mid i_1 > \ldots > i_r\}$. The set $\{\wedge_q u_\mathbf{i} \mid \mathbf{i} \in \mathbb{Z}^r_>\}$ of ordered wedges is a basis of the vector space $\wedge^r_q U$.*

Proof — By Proposition 8, the $\wedge_q u_\mathbf{i}$ with $i_1 > i_2 > \cdots > i_r$ span $\wedge^r_q U$. To prove that they are linearly independent, we consider the q-antisymmetrizer

$$\alpha_r := \sum_{s \in \mathfrak{S}_r} (-q)^{l(s) - l(w_0)} T_s.$$

Using the relation

$$T_s T_k = \begin{cases} T_{ss_k} & \text{if } l(ss_k) = l(s) + 1, \\ T_{ss_k} + (q^{-1} - q) T_s & \text{if } l(ss_k) > l(s) - 1, \end{cases}$$

it is easy to check that for every k we have $\alpha_r (T_k + q) = 0$. Suppose that $\sum_\mathbf{i} a_\mathbf{i} (\wedge_q u_\mathbf{i}) = 0$ for some scalars $a_\mathbf{i} \in \mathbb{Q}(q)$. This means that $\sum_\mathbf{i} a_\mathbf{i} u_\mathbf{i} \in \mathscr{I}_r \subset \ker \alpha_r$, thus $\sum_\mathbf{i} a_\mathbf{i} \alpha_r u_\mathbf{i} = 0$. We are therefore reduced to prove that the tensors $\alpha_r u_\mathbf{i}$ with $i_1 > i_2 > \cdots > i_r$ are linearly independent. Since they belong to $\oplus_{\mathbf{j} \in \mathbb{Z}^r} \mathbb{Z}[q, q^{-1}] u_\mathbf{j}$, we can specialize q to 1, and it is then classical that these specialized tensors are linearly independent over $\mathbb{Z}$. □

3.6 The bar involution of $\wedge^r_q U$

Note that $\overline{T_i + q} = T_i + q$, hence the bar involution of $U^{\otimes r}$ preserves $\mathscr{I}_r$ and one can define a semi-linear involution on $\wedge^r_q U$ by

$$\overline{\mathrm{pr}\,(u)} = \mathrm{pr}\,(\overline{u}), \qquad (u \in U^{\otimes r}).$$

Proposition 10 *Let $\mathbf{i} \in A_r$ and $\mathbf{j} \in \mathbf{i} \cdot \widehat{\mathfrak{S}}_r$. Then*

$$\overline{\wedge_q u_\mathbf{j}} = (-1)^{l(w_0)} q^{l(w_0) - l(w_{0,\mathbf{i}})} \wedge_q u_{w_0 \mathbf{j}}.$$

Proof — For any $u_\mathbf{j} \in U^{\otimes r}$ and $k = 1, \ldots, r-1$, we have

$$(q^{-1} + T_k^{-1}) u_\mathbf{j} = (T_k + q) u_\mathbf{j} \in \mathscr{I}_r,$$

hence $\mathrm{pr}\,(T_k^{-1} u_\mathbf{j}) = -q^{-1} \mathrm{pr}\,(u_\mathbf{j})$. It follows that $\mathrm{pr}\,(T_{w_0}^{-1} u_\mathbf{j}) = (-q)^{-l(w_0)} \mathrm{pr}\,(u_\mathbf{j})$. By Proposition 7 we have

$$\overline{\wedge_q u_\mathbf{j}} = q^{l(w_0)} \mathrm{pr}\,(\overline{u_\mathbf{j}}) = q^{l(w_0) - l(w_{0,\mathbf{i}})}\, \mathrm{pr}\,(T_{w_0}^{-1} u_{w_0 \mathbf{j}}) = (-1)^{l(w_0)} q^{l(w_0) - l(w_{0,\mathbf{i}})} \wedge_q u_{w_0 \mathbf{j}}.$$

□

Proposition 10 allows to compute the expansion of $\overline{\wedge_q u_\mathbf{j}}$ on the basis of ordered wedges by using the straightening algorithm of Proposition 8.

Exercise 12 Take $r = 2$ and $n = 2$. Check that

$$\overline{u_4 \wedge_q u_1} = u_4 \wedge_q u_1 + (q - q^{-1})\, u_3 \wedge_q u_2.$$

For $\mathbf{i} \in \mathbb{Z}^r_>$ write $\overline{\wedge_q u_\mathbf{i}} = \sum_{\mathbf{j} \in \mathbb{Z}^r_>} a_{\mathbf{ij}}(q)\,(\wedge_q u_\mathbf{j})$. Using Proposition 10 and Proposition 8, we easily see that the coefficients $a_{\mathbf{ij}}(q) \in \mathbb{Z}[q, q^{-1}]$ satisfy the following properties.

Proposition 11 (i) *The coefficients $a_{\mathbf{ij}}(q)$ are invariant under translation of* $\mathbf{i}$ *and* $\mathbf{j}$ *by* $(1, \ldots, 1)$. *Hence, setting $\rho = (r-1, r-2, \ldots, 1, 0)$, it is enough to describe the $a_{\mathbf{ij}}(q)$ for which $\mathbf{i} - \rho$ and $\mathbf{j} - \rho$ have non-negative components,* i.e.*for which $\mathbf{i} - \rho$ and $\mathbf{j} - \rho$ are partitions.*

(ii) *If $a_{\mathbf{ij}}(q) \neq 0$ then $\mathbf{i} \in \widetilde{\mathfrak{S}}_r \mathbf{j}$. In particular, if $\mathbf{i} - \rho$ and $\mathbf{j} - \rho$ are partitions, they are partitions of the same integer k.*

(iii) *The matrix $\mathbf{A}_k$ with entries the $a_{\mathbf{ij}}(q)$ for which $\mathbf{i} - \rho$ and $\mathbf{j} - \rho$ are partitions of k is lower unitriangular if the columns and rows are indexed in decreasing lexicographic order.* $\square$

Exercise 13 For $n = 2$ and $r = 3$, check that the matrices $\mathbf{A}_k$ for $k = 2, 3, 4$ are

$$\begin{array}{c} \begin{array}{cc} (4,1,0) & (3,2,0) \end{array} \\ \begin{array}{cc} 1 & 0 \\ q - q^{-1} & 1 \end{array} \end{array} \qquad \begin{array}{c} \begin{array}{ccc} (5,1,0) & (4,2,0) & (3,2,1) \end{array} \\ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ q - q^{-1} & 0 & 1 \end{array} \end{array} \qquad \begin{array}{c} \begin{array}{cccc} (6,1,0) & (5,2,0) & (4,3,0) & (4,2,1) \end{array} \\ \begin{array}{cccc} 1 & 0 & 0 & 0 \\ q - q^{-1} & 1 & 0 & 0 \\ q^{-2} - 1 & q - q^{-1} & 1 & 0 \\ 0 & q^2 - 1 & q - q^{-1} & 1 \end{array} \end{array}$$

3.7 Canonical bases of $\wedge^r_q U$

Let L^+ (resp. L^-) be the $\mathbb{Z}[q]$ (*resp.* $\mathbb{Z}[q^{-1}]$)-lattice in $\wedge^r_q U$ with basis $\{\wedge_q u_\mathbf{i} \mid \mathbf{i} \in \mathbb{Z}^r_>\}$. The fact that the matrix of the bar involution is unitriangular on the basis $\{\wedge_q u_\mathbf{i} \mid \mathbf{i} \in \mathbb{Z}^r_>\}$ implies, by a classical argument going back to Kazhdan and Lusztig, that

Theorem 2 *There exist bases $B^+ = \{G^+_\mathbf{i} \mid \mathbf{i} \in \mathbb{Z}^r_>\}$, $B^- = \{G^-_\mathbf{i} \mid \mathbf{i} \in \mathbb{Z}^r_>\}$ of $\wedge^r_q U$ characterized by:*

(i) $\overline{G^+_\mathbf{i}} = G^+_\mathbf{i}, \quad \overline{G^-_\mathbf{i}} = G^-_\mathbf{i},$

(ii) $G^+_\mathbf{i} \equiv \wedge_q u_\mathbf{i} \bmod qL^+, \quad G^-_\mathbf{i} \equiv \wedge_q u_\mathbf{i} \bmod q^{-1}L^-.$

Proof — Let us prove the existence of B^+. Fix an integer $k \geqslant 0$, and consider the subspace U_k of $\wedge^r_q U$ spanned by the $\wedge_q u_\mathbf{i}$ with $\mathbf{i} - \rho$ a partition of k. Let $I_k = \{\mathbf{i}_1, \mathbf{i}_2, \ldots, \mathbf{i}_m\}$ be the list of $\mathbf{i} \in \mathbb{Z}^r_>$ such that $\mathbf{i} - \rho$ is a partition of k, arranged in decreasing lexicographic order. By Proposition 11 (i) (ii), it is enough to prove that there exists a basis $\{G^+_\mathbf{i}\}$ of U_k indexed by I_k and satisfying the two conditions of the theorem. By Proposition 11 (iii), $\overline{\wedge_q u_{\mathbf{i}_m}} = \wedge_q u_{\mathbf{i}_m}$, so we can take $G_{\mathbf{i}_m} = u_{\mathbf{i}_m}$. We now argue by induction and suppose that for a certain $r < m$ we have constructed vectors $G^+_{\mathbf{i}_{r+1}}, G^+_{\mathbf{i}_{r+2}}, \ldots, G^+_{\mathbf{i}_m}$ satisfying the conditions of the theorem. Moreover, we assume that

$$G^+_{\mathbf{i}_{r+i}} = \wedge_q u_{\mathbf{i}_{r+i}} + \sum_{i < j \leqslant m-r} \alpha_{ij}(q) \wedge_q u_{\mathbf{i}_{r+j}}\,, \qquad (i = 1, \ldots, m-r), \tag{34}$$

for some coefficients $\alpha_{ij}(q) \in \mathbb{Z}[q]$. In other words, we make the additional assumption that the expansion of $G^+_{\mathbf{i}_{r+i}}$ only involves vectors $\wedge_q u_\mathbf{j}$ with $\mathbf{j} \leqslant \mathbf{i}_{r+i}$. We can therefore write, by solving a linear system with unitriangular matrix,

$$\overline{\wedge_q u_{\mathbf{i}_r}} = \wedge_q u_{\mathbf{i}_r} + \sum_{1 \leqslant j \leqslant m-r} \beta_j(q)\, G^+_{\mathbf{i}_{r+j}}\,,$$

where the coefficients $\beta_j(q)$ belong to $\mathbb{Z}[q,q^{-1}]$. By applying the bar involution to this equation we get that $\beta_j(q^{-1}) = -\beta_j(q)$, hence $\beta_j(q) = \gamma_j(q) - \gamma_j(q^{-1})$ with $\gamma_j(q) \in q\mathbb{Z}[q]$. Now set

$$G^+_{\mathbf{i}_r} = \wedge_q u_{\mathbf{i}_r} + \sum_{1\leqslant j\leqslant m-r} \gamma_j(q)\, G^+_{\mathbf{i}_{r+j}}\,.$$

We have $G^+_{\mathbf{i}_r} \equiv \wedge_q u_{\mathbf{i}_r} \bmod qL^+$, $\overline{G^+_{\mathbf{i}_r}} = G^+_{\mathbf{i}_r}$, and the expansion of $G(\lambda^r)$ on the standard basis is of the form (34) as required, hence the existence of B^+ follows by induction.

The proof of the existence of B^- is similar.

To prove unicity, we show that if $x \in qL^+$ is bar-invariant then $x = 0$. Otherwise write $x = \sum_{\mathbf{i}} \theta_{\mathbf{i}}(q) \wedge_q u_{\mathbf{i}}$, and let $\mathbf{j}$ be maximal such that $\theta_{\mathbf{j}}(q) \neq 0$. Then $\wedge_q u_{\mathbf{j}}$ occurs in $\bar{x}$ with coefficient $\theta_{\mathbf{j}}(q^{-1})$, hence $\theta_{\mathbf{j}}(q) = \theta_{\mathbf{j}}(q^{-1})$. But since $\theta_{\mathbf{j}}(q) \in q\mathbb{Z}[q]$ this is impossible. $\square$

Set

$$G^+_{\mathbf{j}} = \sum_{\mathbf{i}} c_{\mathbf{ij}}(q)\,(\wedge_q u_{\mathbf{i}})\,, \qquad G^-_{\mathbf{i}} = \sum_{\mathbf{j}} l_{\mathbf{ij}}(-q^{-1})\,(\wedge_q u_{\mathbf{j}})\,.$$

Let $\mathbf{C}_k$ and $\mathbf{L}_k$ denote respectively the matrices with entries the coefficients $c_{\mathbf{ij}}(q)$ and $l_{\mathbf{ij}}(q)$ for which $\mathbf{i}-\rho$ and $\mathbf{j}-\rho$ are partitions of k.

Exercise 14 For $r = 3$ and $n = 2$, check that we have

$$\mathbf{C}_4 = \begin{array}{cccc} (6,1,0) & (5,2,0) & (4,3,0) & (4,2,1) \\ 1 & 0 & 0 & 0 \\ q & 1 & 0 & 0 \\ 0 & q & 1 & 0 \\ q & q^2 & q & 1 \end{array} \qquad \mathbf{L}_4 = \begin{array}{cccc} (6,1,0) & (5,2,0) & (4,3,0) & (4,2,1) \\ 1 & q & q^2 & 0 \\ 0 & 1 & q & 0 \\ 0 & 0 & 1 & q \\ 0 & 0 & 0 & 1 \end{array}$$

3.8 Action of $U_q(\widehat{\mathfrak{g}})$ and $Z(\widehat{H}_r)$ on $\wedge_q^r U$

Since the action of $U_q(\widehat{\mathfrak{g}})$ on $U^{\otimes r}$ commutes with the action of $\widehat{H}_r$, the subspace $\mathscr{I}_r$ is stable under $U_q(\widehat{\mathfrak{g}})$ and we obtain an induced action of $U_q(\widehat{\mathfrak{g}})$ on $\wedge_q^r U$. The action on $\wedge_q u_{\mathbf{i}}$ of the generators of $U_q(\widehat{\mathfrak{sl}}_n)$ is obtained by projecting (23), (24):

$$f_k(\wedge_q u_{\mathbf{i}}) \;=\; \sum_{\substack{j=1\\ i_j\equiv k}}^{r} q^{\sum_{l=1}^{j-1}(\delta_{i_l\equiv k}-\delta_{i_l\equiv k+1})}\,(\wedge_q u_{\mathbf{i}+\varepsilon_j}), \tag{35}$$

$$e_k(\wedge_q u_{\mathbf{i}}) \;=\; \sum_{\substack{j=1\\ i_j\equiv k+1}}^{r} q^{-\sum_{l=j+1}^{r}(\delta_{i_l\equiv k}-\delta_{i_l\equiv k+1})}\,(\wedge_q u_{\mathbf{i}-\varepsilon_j}). \tag{36}$$

Note that if $\mathbf{i} \in \mathbb{Z}^r_>$ then $\mathbf{i}\pm\varepsilon_j \in \mathbb{Z}^r_\geqslant := \{\mathbf{j}\in\mathbb{Z}^r \mid j_1 \geqslant \cdots \geqslant j_r\}$. It follows that either $\wedge_q u_{\mathbf{i}\pm\varepsilon_j}$ belongs to the basis $\{\wedge_q u_{\mathbf{j}} \mid \mathbf{j}\in\mathbb{Z}^r_>\}$, or $\wedge_q u_{\mathbf{i}\pm\varepsilon_j} = 0$. Hence, Eq. (35) (36) require no straightening relation and are very simple to use in practice.

By a classical result of Bernstein (see [**Lu1**, Th. 8.1]), the center $Z(\widehat{H}_r)$ of $\widehat{H}_r$ is the algebra of symmetric Laurent polynomials in the elements Y_i. Clearly, $Z(\widehat{H}_r)$ leaves invariant the submodule $\mathscr{I}_r$. It follows that $Z(\widehat{H}_r)$ acts on $\wedge_q^r U = U^{\otimes r}/\mathscr{I}_r$. This action can be computed via (31). In particular $B_k = \sum_{i=1}^r Y_i^k$ acts by

$$B_k\,(\wedge_q u_{\mathbf{i}}) = \sum_{j=1}^{r} \wedge_q u_{\mathbf{i}-nk\varepsilon_j}, \qquad (k\in\mathbb{Z}^*). \tag{37}$$

Note that the right-hand side of (37) may involve terms $\wedge_q u_{\mathbf{j}}$ with $\mathbf{j} \notin \mathbb{Z}^r_{\geqslant}$ which have to be expressed on the basis $\{\wedge_q u_{\mathbf{i}} \mid \mathbf{i} \in \mathbb{Z}^r_{\geqslant}\}$ by repeated applications of the straightening rules.

Example 2 Take $r = 4$ and $n = 2$. We have

$$B_{-2}\left(\wedge_q u_{(3,2,1,0)}\right) = \wedge_q u_{(7,2,1,0)} + \wedge_q u_{(3,6,1,0)} + \wedge_q u_{(3,2,5,0)} + \wedge_q u_{(3,2,1,4)}.$$

By Proposition 8,

$$\begin{aligned}
\wedge_q u_{(3,6,1,0)} &= -q^{-1} \wedge_q u_{(6,3,1,0)} + (q^{-2}-1) \wedge_q u_{(5,4,1,0)}, \\
\wedge_q u_{(3,2,5,0)} &= -q^{-1} \wedge_q u_{(3,5,2,0)} + (q^{-2}-1) \wedge_q u_{(3,4,3,0)} = q^{-1} \wedge_q u_{(5,3,2,0)}, \\
\wedge_q u_{(3,2,1,4)} &= -q^{-1} \wedge_q u_{(3,2,4,1)} + (q^{-2}-1) \wedge_q u_{(3,2,3,2)} = -q^{-2} \wedge_q u_{(4,3,2,1)},
\end{aligned}$$

which yields

$$\begin{aligned}
B_{-2}\left(\wedge_q u_{(3,2,1,0)}\right) = \wedge_q u_{(7,2,1,0)} &- q^{-1} \wedge_q u_{(6,3,1,0)} \\
&+ (q^{-2}-1) \wedge_q u_{(5,4,1,0)} + q^{-1} \wedge_q u_{(5,3,2,0)} - q^{-2} \wedge_q u_{(4,3,2,1)}.
\end{aligned}$$

3.9 The Fock space

As in §2.2, we shall now construct the Fock space representation of $U_q(\widehat{\mathfrak{g}})$ as the direct limit of vector spaces

$$\mathscr{F} = \lim_{\rightarrow} \wedge_q^r U$$

with respect to the linear maps $\varphi_{r,s} : \wedge_q^r U \to \wedge_q^s U$ $(r \leqslant s)$ defined by

$$\varphi_{r,s}(u_{i_1} \wedge_q \cdots \wedge_q u_{i_r}) = u_{i_1} \wedge_q \cdots \wedge_q u_{i_r} \wedge_q u_{-r} \wedge_q u_{-r-1} \wedge_q \cdots \wedge_q u_{-s+1}.$$

By construction, each $\wedge_q u_{\mathbf{i}} \in \wedge_q^r U$ has an image $\varphi_r(\wedge_q u_{\mathbf{i}}) \in \mathscr{F}$ which we think of as the infinite q-wedge

$$\varphi_r(\wedge_q u_{\mathbf{i}}) = u_{i_1} \wedge_q \cdots \wedge_q u_{i_r} \wedge_q u_{-r} \wedge_q u_{-r-1} \wedge_q \cdots \wedge_q u_{-s+1} \wedge_q \cdots$$

Given a partition $\lambda = (\lambda_i)_{i \in \mathbb{N}}$, where we assume as before that $\lambda_i = 0$ for i large enough, we set

$$|\lambda\rangle := u_{i_1} \wedge_q u_{i_2} \wedge_q \cdots \wedge_q u_{i_r} \wedge \cdots,$$

where $i_k = \lambda_k - k + 1$ $(k \geqslant 1)$. It follows from Proposition 8 and Proposition 9 that $\{|\lambda\rangle \mid \lambda \in \mathscr{P}\}$ is a basis of $\mathscr{F}$. As in §2.2.1, we will be using the natural grading of $\mathscr{F}$ given by

$$\deg(|\lambda\rangle) = \sum_k \lambda_k.$$

We will sometimes write $|\lambda|$ instead of $\deg(|\lambda\rangle)$, and $\lambda \vdash k$ if $|\lambda| = k$.

3.10 Action of $U_q(\widehat{\mathfrak{g}})$ on $\mathscr{F}$

As in §2.2.4 and §2.2.5, when $r \to \infty$ the compatible actions of $U_q(\widehat{\mathfrak{g}})$ on the q-wedge spaces $\wedge_q^r U$ give an action on $\mathscr{F}$, by setting

$$f_i \varphi_r(\wedge_q u_{\mathbf{i}}) := \varphi_{r+1} f_i \varphi_{r,r+1}(\wedge_q u_{\mathbf{i}}), \quad e_i \varphi_r(\wedge_q u_{\mathbf{i}}) := q^{-\delta_{i \equiv r}} \varphi_r(e_i(\wedge_q u_{\mathbf{i}})), \qquad (\mathbf{i} \in \mathbb{Z}^r_{>}).$$

These formulas have a nice combinatorial description in terms of Young diagrams. Given two partitions λ and μ such that the Young diagram of μ is obtained by adding an i-cell γ to the

Young diagram of λ, let $A_i^r(\lambda,\mu)$ (*resp.* $R_i^r(\lambda,\mu)$) be the number of addable i-cells of λ (*resp.* of removable i-cells of λ) situated to the right of γ (γ not included). Set

$$N_i^r(\lambda,\mu) = A_i^r(\lambda,\mu) - R_i^r(\lambda,\mu).$$

Then Eq. (35) gives

$$f_i|\lambda\rangle = \sum_\mu q^{N_i^r(\lambda,\mu)}|\mu\rangle, \tag{38}$$

where the sum is over all partitions μ obtained from λ by adding an i-cell. Similarly, Eq. (36) gives

$$e_i|\mu\rangle = \sum_\lambda q^{-N_i^l(\alpha,\beta)}|\lambda\rangle, \tag{39}$$

where the sum is over all partitions λ obtained from μ by removing an i-cell, and $N_i^l(\lambda,\mu)$ is defined as $N_i^r(\lambda,\mu)$ but replacing right by left.

Exercise 15 Check that, for $n=2$, we have

$$\begin{array}{rclcrcl} f_0|3,1\rangle &=& q^{-1}|3,2\rangle + |3,1,1\rangle, & \qquad & f_1|3,1\rangle &=& |4,1\rangle, \\ e_0|3,1\rangle &=& q^{-2}|2,1\rangle, & & e_1|3,1\rangle &=& |3\rangle. \end{array}$$

Exercise 16 Show that $t_i|\lambda\rangle = q^{N_i(\lambda)}|\lambda\rangle$, where $N_i(\lambda)$ denotes the number of addable i-nodes minus the number of removable i-nodes of λ. Deduce that the central element $K = t_0\cdots t_{n-1}$ acts on $\mathscr{F}$ as $q\,\mathrm{Id}_{\mathscr{F}}$, *i.e.* $\mathscr{F}$ is a level one representation of $U_q(\widehat{\mathfrak{g}})$.

3.11 Action of the bosons on $\mathscr{F}$

Let $\mathbf{i} \in \mathbb{Z}^r_>$. It follows from the easily checked relations

$$u_{-s}\wedge_q u_{-r}\wedge_q u_{-r-1}\wedge_q\cdots\wedge_q u_{-s} = 0, \quad u_{-r}\wedge_q u_{-r-1}\wedge_q\cdots\wedge_q u_{-s}\wedge_q u_{-r} = 0, \quad (s \geqslant r \geqslant 0)$$

that the vector $\varphi_s B_k \varphi_{r,s}(\wedge_q u_{\mathbf{i}})$ is independent of s for $s > r$ large enough. Hence one can define endomorphisms B_k of $\mathscr{F}$ by

$$B_k\varphi_r(\wedge_q u_{\mathbf{i}}) := \varphi_s B_k \varphi_{r,s}(\wedge_q u_{\mathbf{i}}), \qquad (k\in\mathbb{Z}^*,\ s \gg 1). \tag{40}$$

By construction, these endomorphisms commute with the action of $U_q(\widehat{\mathfrak{g}})$ on $\mathscr{F}$. However they no longer generate a commutative algebra. Using arguments very similar to those of the proof of Proposition 1, Kashiwara, Miwa and Stern [**KMS**] showed that

$$[B_k,B_l] = \begin{cases} k\dfrac{1-q^{-2nk}}{1-q^{-2k}} & \text{if } k=-l, \\ 0 & \text{otherwise.} \end{cases} \tag{41}$$

Hence the B_k generate a Heisenberg algebra that we shall denote by $\mathscr{H}$.

Remark 2 The B_k's are q-analogues of the endomorphisms β_{nk} of §2.2.7. We do not have natural q-analogues of the other bosons β_l with l not a multiple of n. In the classical case, these β_l belong in fact to $\widehat{\mathfrak{g}}$. (For example, $\beta_1 = \sum_i e_i$.) They generate the principal Heisenberg subalgebra $\mathfrak{p}$ of $\widehat{\mathfrak{g}}$. We lack a nice quantum analogue of this principal subalgebra.

Let $\mathbb{C}[\mathscr{H}^-]$ denote the commutative subalgebra of $U(\mathscr{H})$ generated by the B_k $(k<0)$. Let $V(\Lambda_0)$ denote the irreducible highest weight $U_q(\widehat{\mathfrak{g}})$-module with highest weight Λ_0. Using characters and arguing as in §2.2.8, we get the following analogue of Proposition 2.

Proposition 12 *The $U_q(\widehat{\mathfrak{g}})\otimes U(\mathscr{H})$-modules $\mathscr{F}$ and $V(\Lambda_0)\otimes\mathbb{C}[\mathscr{H}^-]$ are isomorphic.* □

3.12 The bar involution of $\mathscr{F}$

Using Proposition 8 we can check the following

Lemma 1 *Let* $\mathbf{i} \in \mathbb{Z}^r$ *and let* $m \geqslant r$. *Assume that* $i_k > -m$ $(k = 1, \ldots, r)$ *and* $\sum_k (i_k + k - 1) \leqslant m$. *Then*

$$u_{-m} \wedge_q u_{i_1} \wedge_q \cdots \wedge_q u_{i_r} \wedge_q u_{-r} \wedge_q \cdots \wedge_q u_{-m+1} \; = \\ (-1)^m q^{-a(\mathbf{i})} u_{i_1} \wedge_q \ldots \wedge_q u_{i_r} \wedge_q u_{-r} \wedge_q \cdots \wedge_q u_{-m+1} \wedge_q u_{-m},$$

where $a(\mathbf{i}) = \sharp\{ j \leqslant r \mid i_j \not\equiv -m\} + \sharp\{-r \geqslant j \geqslant -m+1 \mid j \not\equiv -m\}$. □

Repeated applications of this lemma together with Proposition 10 yield that if $\mathbf{i}$ satisfies the hypothesis of the lemma and $p \geqslant m$, we have

$$\overline{\varphi_{r,p}(\wedge_q u_{\mathbf{i}})} = \varphi_{m,p}(\overline{\varphi_{r,m}(\wedge_q u_{\mathbf{i}})}).$$

Thus we can define a semi-linear involution on $\mathscr{F}$ by putting

$$\overline{\varphi_r(\wedge_q u_{\mathbf{i}})} := \varphi_m(\overline{\varphi_{r,m}(\wedge_q u_{\mathbf{i}})}), \qquad (\mathbf{i} \in \mathbb{Z}^r,\ \deg \varphi_r(\wedge_q u_{\mathbf{i}}) = m,\ i_k > -m \text{ for all } k). \tag{42}$$

In particular, for $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathscr{P}$ and $s \geqslant \sum_k \lambda_k$, we have

$$\overline{|\lambda\rangle} = \varphi_s(\overline{u_{\lambda_1} \wedge_q u_{\lambda_2 - 1} \wedge_q \cdots \wedge_q u_{\lambda_r - r + 1} \wedge_q u_{-r} \wedge_q \cdots \wedge_q u_{-s+1}}).$$

The following proposition shows that the lowering operators of $U_q(\widehat{\mathfrak{g}})$ and $\mathscr{H}$ commute with the bar involution.

Proposition 13 *For* $\lambda \in \mathscr{P}$, $0 \leqslant i \leqslant n-1$ *and* $k \in \mathbb{N}^*$, *we have*

$$\overline{f_i |\lambda\rangle} = f_i \overline{|\lambda\rangle}, \qquad \overline{B_{-k} |\lambda\rangle} = B_{-k} \overline{|\lambda\rangle}.$$

Proof — This readily follows from Eq. (38) (40) (42). (Note that the condition $\lambda_i > -m$ in (42) is preserved by the action of these lowering operators.) □

Because of Proposition 12, it is easy to see that Proposition 13 and the normalization condition $\overline{|\emptyset\rangle} = |\emptyset\rangle$ characterize the bar involution of $\mathscr{F}$. One can also develop a straightening free algorithm based on Proposition 13 for computing the bar involution, which is much more efficient in practice (see [**L2**]).

3.13 Canonical bases of $\mathscr{F}$

Let $\rho_r := (r-1, r-2, \ldots, 1, 0) \in \mathbb{Z}^r_>$. For $\mu \in \mathscr{P}$ write

$$\overline{|\mu\rangle} = \sum_{\lambda \in \mathscr{P}} b_{\lambda\mu}(q) |\lambda\rangle.$$

Then, for $|\lambda| = |\mu| \leqslant r$ it follows from (42) that we have

$$b_{\lambda\mu}(q) = a_{\mathbf{i}\mathbf{j}}(q)$$

where $\mathbf{i} = \lambda + \rho_r$, $\mathbf{j} = \mu + \rho_r$, and the coefficients $a_{\mathbf{i}\mathbf{j}}(q)$ have been defined in § 3.6. Hence by Proposition 11 the matrix

$$\mathbf{B}_k := [b_{\lambda\mu}(q)], \qquad (\lambda, \mu \vdash k)$$

is unitriangular, and one can define canonical bases $\{G^+_\lambda \mid \lambda \in \mathscr{P}\}$, $\{G^-_\lambda \mid \lambda \in \mathscr{P}\}$ of $\mathscr{F}$ characterized by:

(i) $\overline{G^+_\lambda} = G^+_\lambda, \quad \overline{G^-_\lambda} = G^-_\lambda,$

(ii) $G^+_\lambda \equiv |\lambda\rangle \bmod qL^+_\infty, \quad G^-_\lambda \equiv |\lambda\rangle \bmod q^{-1}L^-_\infty,$

where L^+_∞ (*resp.* L^-_∞) is the $\mathbb{Z}[q]$-submodule (*resp.* $\mathbb{Z}[q^{-1}]$-submodule) of $\mathscr{F}$ spanned by the elements $|\lambda\rangle$ of the standard basis. Set

$$G^+_\mu = \sum_\lambda d_{\lambda\mu}(q)\,|\lambda\rangle\,, \qquad G^-_\lambda = \sum_\mu e_{\lambda\mu}(-q^{-1})\,|\mu\rangle\,,$$

and

$$\mathbf{D}_k := [d_{\lambda\mu}(q)], \qquad \mathbf{E}_k := [e_{\lambda\mu}(q)], \qquad (\lambda,\mu \vdash k).$$

Then, for $r \geqslant k$ we have

$$d_{\lambda\mu}(q) = c_{\lambda+\rho_r,\mu+\rho_r}(q), \qquad e_{\lambda\mu}(q) = l_{\lambda+\rho_r,\mu+\rho_r}(q),$$

where the polynomials $c_{\mathbf{ij}}(q)$ and $l_{\mathbf{ij}}(q)$ are those of §3.7.

Exercise 17 Check that

$$\begin{array}{ccccc} (4) & (3,1) & (2,2) & (2,1,1) & (1,1,1,1) \end{array}$$

$$\mathbf{D}_4 = \begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ q & 1 & 0 & 0 & 0 \\ 0 & q & 1 & 0 & 0 \\ q & q^2 & q & 1 & 0 \\ q^2 & 0 & 0 & q & 1 \end{array}$$

Compare with the matrix $\mathbf{C}_4$ of Exercise 14.

3.14 Crystal and global bases

Lusztig's theory of canonical bases for quantum enveloping algebras was inspired by the Kazhdan-Lusztig bases of Iwahori-Hecke algebras. Independently of Lusztig, Kashiwara gave a different construction of canonical bases inspired by works of the Kyoto group on solvable models in statistical mechanics. Since in the framework of statistical mechanics q represents the temperature, one might expect that the representation theory becomes drastically simpler as the temperature tends to absolute zero and the model crystallizes. This was first observed by Date, Jimbo and Miwa for $\mathfrak{g} = \mathfrak{gl}_n$ [**DJM**], and then generalized to an arbitrary symmetrizable Kac-Moody algebra $\mathfrak{g}$ by Kashiwara [**Kas1**].

In Kashiwara's approach, the canonical basis of a $U_q(\mathfrak{g})$-module M, called *global basis* of M, is obtained in two steps. First, one constructs a combinatorial skeleton, called the *crystal basis*, which one should think of as a "basis of M at $q = 0$". The second step involves the same bar involution of $U_q(\mathfrak{g})$ as in Lusztig's construction, which induces a bar involution on every highest weight irreducible $U_q(\mathfrak{g})$-module. Kashiwara shows that for any irreducible integrable highest weight $U_q(\mathfrak{g})$-module M there is a unique $\mathbb{Q}(q)$-basis of M which is bar-invariant and "coincides with the crystal basis at $q = 0$" [**Kas1**]. This is the *global basis* of M, and it is identical to Lusztig's canonical basis of M, as shown by Grojnowski and Lusztig [**GL**].

By Proposition 13, the restriction of the bar involution to the submodule $U_q(\widehat{\mathfrak{g}})|\emptyset\rangle \cong V(\Lambda_0)$ coincides with the Kashiwara-Lusztig bar involution of the irreducible module $V(\Lambda_0)$. Moreover, it was shown by Misra and Miwa [**MM**] that in the crystal limit $q \to 0$ the standard basis of $\mathscr{F}$ tends to a crystal basis of $\mathscr{F}$. The subset of the standard basis labelled by the set of *n-regular partitions*, *i.e.* partitions λ with no part repeated more than $n-1$ times, turns out to give "at $q = 0$" a basis of $V(\Lambda_0)$. This implies

Proposition 14 *The subset* $\{G^+(\lambda) \mid \lambda$ *is n-regular*$\}$ *coincides with Kashiwara's global basis (or Lusztig's canonical basis) of the basic representation* $V(\Lambda_0)$. □

Thus we have obtained a global basis $\{G^+(\lambda) \mid \lambda \in \mathscr{P}\}$ of the Fock space extending the global basis of its highest irreducible $U_q(\widehat{\mathfrak{g}})$-submodule. In [**Kas2**], Kashiwara has generalized these results to the more general Fock space representations of quantum affine algebras introduced in [**KMPY**].

4 Decomposition numbers

4.1 The Lusztig character formula

Let $U_v(\mathfrak{gl}_r)$ be the quantum enveloping algebra of $\mathfrak{gl}_r$. This is a $\mathbb{Q}(v)$-algebra with generators $e_i,\ f_i,\ K_j\ (1 \leqslant i \leqslant r-1,\ 1 \leqslant j \leqslant r)$ subject to the relations

$$K_jK_j^{-1} = K_j^{-1}K_j = 1, \qquad K_jK_l = K_lK_j$$

$$K_je_iK_j^{-1} = v^{\delta_{ij}-\delta_{j,i+1}}e_j, \qquad K_jf_iK_j^{-1} = v^{-\delta_{ij}+\delta_{j,i+1}}f_j, \qquad e_if_j - e_jf_i = \delta_{ij}\frac{K_iK_{i+1}^{-1} - K_i^{-1}K_{i+1}}{v-v^{-1}},$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_v e_i^{1-a_{ij}-k}e_je_i^k = 0 \quad (i \neq j),$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_v f_i^{1-a_{ij}-k}f_jf_i^k = 0 \quad (i \neq j).$$

Here, $A = [a_{ij}]_{1\leqslant i,j\leqslant r-1}$ is the Cartan matrix of type A_{r-1}. Let $U_{v,\mathbb{Z}}(\mathfrak{gl}_r)$ denote the $\mathbb{Z}[v,v^{-1}]$-subalgebra generated by the elements

$$e_i^{(k)} := \frac{e_i^k}{[k]!}, \quad f_i^{(k)} := \frac{f_i^k}{[k]!}, \quad K_j^{\pm}, \qquad (k \in \mathbb{N}).$$

Let $\zeta \in \mathbb{C}$ be such that ζ^2 is a primitive nth root of 1. One defines $U_\zeta(\mathfrak{gl}_r) := U_{v,\mathbb{Z}}(\mathfrak{gl}_r) \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbb{C}$ where $\mathbb{Z}[v,v^{-1}]$ acts on $\mathbb{C}$ by $v \mapsto \zeta$ [**Lu2**, **Lu3**].

Let $\lambda \in \mathbb{Z}^r_{\geqslant}$. There is a unique finite-dimensional $U_v(\mathfrak{gl}_r)$-module (of type 1) $W_v(\lambda)$ with highest weight λ. Its character is the same as for $\mathfrak{gl}_r$ and is given by Weyl's character formula. Fix a highest weight vector $u_\lambda \in W_v(\lambda)$ and denote by $W_{v,\mathbb{Z}}(\lambda)$ the $U_{v,\mathbb{Z}}(\mathfrak{gl}_r)$-submodule of $W_v(\lambda)$ generated by u_λ. Finally, put

$$W_\zeta(\lambda) := W_{v,\mathbb{Z}}(\lambda) \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbb{C}.$$

This is a $U_\zeta(\mathfrak{gl}_r)$-module called a Weyl module [**Lu2**]. By definition $\mathrm{ch}W_\zeta(\lambda) = \mathrm{ch}W_v(\lambda)$. There is a unique simple quotient of $W_\zeta(\lambda)$ denoted by $L(\lambda)$. Its character is given in terms of the characters of the Weyl modules by the so-called Lusztig conjecture [**Lu3**] (now a theorem of Kazhdan-Lusztig [**KL**] and Kashiwara-Tanisaki [**KT**]).

To state it, recall the action of $\widehat{\mathfrak{S}}_r$ on $\mathbb{Z}^r$ introduced in §2.1.5. Namely, $s_k\ (1 \leqslant k \leqslant r-1)$ acts by switching the kth and $(k+1)$th components of $\mathbf{i}$, and $z_j\ (1 \leqslant j \leqslant r)$ by translating the jth component by $-n$.

Theorem 3 (Kazhdan-Lusztig [KL], Kashiwara-Tanisaki [KT]) *Let $\rho = (r-1, r-2, \ldots, 1, 0)$. We have*

$$\mathrm{ch}L(\lambda) = \sum_{\mu} P^{-}_{\mu+\rho,\lambda+\rho}(-1)\,\mathrm{ch}W_{\zeta}(\mu), \tag{43}$$

where the sum is over all $\mu \in \mathbb{Z}^r_{\geqslant}$ such that $\mu+\rho$ belongs to the $\widehat{\mathfrak{S}}_r$-orbit of $\lambda+\rho$.

In this formula, the coefficients $P^{-}_{\mu+\rho,\lambda+\rho}(-1)$ are values at $q=-1$ of some parabolic Kazhdan-Lusztig polynomials for $\widehat{\mathfrak{S}}_r$, whose definition will be recalled in the next section.

Example 3 Take $r=3, n=2$ and $\lambda=(4,0,0)$. Then $\lambda+\rho=(6,1,0)$. The only dominant weights $\mu \in \mathbb{Z}^3_{\geqslant}$ such that $P^{-}_{\mu+\rho,(6,1,0)}(q) \neq 0$ are $(4,0,0)$, $(3,1,0)$, and $(2,2,0)$, and one can calculate

$$P_{(6,1,0),(6,1,0)}(q) = 1, \quad P_{(5,2,0),(6,1,0)}(q) = q, \quad P_{(4,3,0),(6,1,0)}(q) = q^2.$$

It follows that the character of $L(4,0,0)$ for $\zeta^2=-1$ is given by

$$\mathrm{ch}L(4,0,0) = \mathrm{ch}W_{\zeta}(4,0,0) - \mathrm{ch}W_{\zeta}(3,1,0) + \mathrm{ch}W_{\zeta}(2,2,0).$$

4.2 Parabolic Kazhdan-Lusztig polynomials

The parabolic versions of the Kazhdan-Lusztig polynomials have been introduced by Deodhar [**D**]. We refer to [**So**] for a more detailed exposition.

Up to now, we have been regarding $\widehat{\mathfrak{S}}_r$ as the semidirect product $\mathfrak{S}_r \ltimes \mathbb{Z}^r$, and we have used the corresponding Bernstein presentation of $\widehat{H}_r$. To introduce the Kazhdan-Lusztig polynomials, we need to shift our point of view and consider $\widehat{\mathfrak{S}}_r$ as an extended Coxeter group, with generators $s_0, s_1, \ldots, s_{r-1}, \tau$ subject to the relations

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \tag{44}$$
$$s_i s_j = s_j s_i, \qquad (i-j \neq \pm 1), \tag{45}$$
$$s_i^2 = 1, \tag{46}$$
$$\tau s_i = s_{i+1}\tau. \tag{47}$$

Here the indices $i = 0, 1, \ldots r-1$ are taken modulo r. The new generators s_0 and τ are expressed in terms of the old ones by

$$\tau = s_1 s_2 \cdots s_{r-1} z_r, \tag{48}$$
$$s_0 = s_{r-1} s_{r-2} \cdots s_2 s_1 s_2 \cdots s_{r-1} z_1^{-1} z_r. \tag{49}$$

The subgroup $\widetilde{\mathfrak{S}}_r$ generated by $s_0, s_1, \ldots, s_{r-1}$ is a Coxeter group of type $\widetilde{A}_{r-1}$, and therefore has an associated Bruhat order, and a length function. But $\widehat{\mathfrak{S}}_r$ is not a Coxeter group. However, one can extend the Bruhat order and the length function of $\widetilde{\mathfrak{S}}_r$ to $\widehat{\mathfrak{S}}_r$ as follows. Let $w = \tau^k \sigma, w' = \tau^m \sigma'$ with $k, m \in \mathbb{Z}$, $\sigma, \sigma' \in \widetilde{\mathfrak{S}}_r$. We say that $w < w'$ if and only if $k = m$ and $\sigma < \sigma'$, and we put $l(w) := l(\sigma)$.

The Hecke algebra $\widehat{H}_r$ has the following alternative description. This is the algebra over $\mathbb{Q}(q)$ with basis T_w ($w \in \widehat{\mathfrak{S}}_r$) and multiplication defined by

$$T_w T_{w'} = T_{ww'} \qquad \text{if } l(ww') = l(w) + l(w'), \tag{50}$$
$$(T_{s_i} - q^{-1})(T_{s_i} + q) = 0, \qquad (0 \leqslant i \leqslant r-1). \tag{51}$$

The generators used in §3.3 are $T_i = T_{s_i}$ $(1 \leqslant i \leqslant r-1)$ and $Y_j = T_{z_j}$ $(1 \leqslant j \leqslant r)$. We can now also use $T_0 = T_{s_0}$ and T_τ, which we shall simply denote by τ. This gives another presentation:

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \tag{52}$$
$$T_i T_j = T_j T_i, \qquad (i-j \neq \pm 1), \tag{53}$$
$$(T_i - q^{-1})(T_i + q) = 0, \tag{54}$$
$$\tau T_i = T_{i+1}\tau, \tag{55}$$

where now the subscripts $i = 0, 1, \ldots, r-1$ are understood modulo r.

For the action of $\widehat{\mathfrak{S}}_r$ on $\mathbb{Z}^r$ defined in §2.1.5, we have

$$s_0 \cdot \mathbf{i} = (i_r - n, i_2, \ldots, i_{r-1}, i_1 + n), \qquad \tau \cdot \mathbf{i} = (i_r - n, i_1, i_2, \ldots, i_{r-1}).$$

We shall now define another basis $(v_\mathbf{i} \mid \mathbf{i} \in \mathbb{Z}^r)$ of $U^{\otimes r}$. Put $v_\mathbf{i} = u_\mathbf{i}$ if $\mathbf{i} \in A_r$. Otherwise, there is a unique $\mathbf{j} \in A_r$ such that $\mathbf{i}$ belongs to the $\widehat{\mathfrak{S}}_r$-orbit of $\mathbf{j}$, and a unique element $w = w(\mathbf{i})$ of minimal length such that $\mathbf{i} = w \cdot \mathbf{j}$. Then put $v_\mathbf{i} = T_w v_\mathbf{j} = T_w u_\mathbf{j}$. This new basis is better adapted to the Coxeter-type presentation of $\widehat{H}_r$. Indeed, putting $i_0 = i_r - n$, it is not difficult to see that

$$\tau v_\mathbf{i} \quad = \quad v_{\mathbf{i}\cdot\tau}, \qquad (\mathbf{i} \in \mathbb{Z}^r), \tag{56}$$

$$T_k v_\mathbf{i} \quad = \quad \begin{cases} v_{s_k \cdot \mathbf{i}} & \text{if } i_k < i_{k+1}, \\ q^{-1} v_\mathbf{i} & \text{if } i_k = i_{k+1}, \\ v_{s_k\cdot\mathbf{i}} + (q^{-1} - q) v_\mathbf{i} & \text{if } i_k > i_{k+1}, \end{cases} \qquad (0 \leqslant k \leqslant r-1,\ \mathbf{i} \in \mathbb{Z}_r). \tag{57}$$

Lemma 2 *For $\mathbf{i} \in \mathbb{Z}^r_{\geqslant}$, we have $v_\mathbf{i} = u_\mathbf{i}$.*

Proof — Write $\mathbf{i} = z^\mathbf{k} s \cdot \mathbf{j}$ with $\mathbf{j} \in A_r$, $\mathbf{k} \in \mathbb{Z}^r$, and $s \in \mathfrak{S}_r$. It is easy to see that the assumption $\mathbf{i} \in \mathbb{Z}^r_{\geqslant}$ forces $\mathbf{k} \in \mathbb{Z}^r_{\leqslant}$. This in turn implies that $l(z^\mathbf{k} s) = l(z^\mathbf{k}) + l(s)$, hence $T_{z^\mathbf{k}} T_s = T_{z^\mathbf{k} s}$. Now for $\mathbf{k} \in \mathbb{Z}^r_{\leqslant}$ one also has $T_{z^\mathbf{k}} = Y^\mathbf{k}$, so $v_\mathbf{i} = T_{z^\mathbf{k} s} u_\mathbf{j} = Y^\mathbf{k}(T_s u_\mathbf{j}) = Y^\mathbf{k} u_{s\cdot\mathbf{j}} = u_\mathbf{i}$. □

Recall the decomposition $U^{\otimes r} = \oplus_{\mathbf{i} \in A_r} \widehat{H}_r u_\mathbf{i}$ of §3.4. For every $\mathbf{i} \in A_r$, the $\widehat{H}_r$-submodule

$$\widehat{H}_r u_\mathbf{i} = \bigoplus_{\mathbf{j} \in \widehat{\mathfrak{S}}_r \mathbf{i}} \mathbb{Q}(q) u_\mathbf{j} = \bigoplus_{\mathbf{j} \in \widehat{\mathfrak{S}}_r \mathbf{i}} \mathbb{Q}(q) v_\mathbf{j}$$

is a parabolic module, and therefore has two Kazhdan-Lusztig bases defined as follows. Consider the two lattices

$$\mathscr{L}_\mathbf{i}^+ = \bigoplus_{\mathbf{j} \in \widehat{\mathfrak{S}}_r \mathbf{i}} \mathbb{Z}[q] v_\mathbf{j}, \qquad \mathscr{L}_\mathbf{i}^- = \bigoplus_{\mathbf{j} \in \widehat{\mathfrak{S}}_r \mathbf{i}} \mathbb{Z}[q^{-1}] v_\mathbf{j}.$$

By work of Deodhar [**D**], there are two bases $C_\mathbf{j}^+$, $C_\mathbf{j}^-$ $(\mathbf{j} \in \widehat{\mathfrak{S}}_r \mathbf{i})$ of $\widehat{H}_r u_\mathbf{i}$ characterized by

$$\overline{C_\mathbf{j}^+} = C_\mathbf{j}^+, \qquad \overline{C_\mathbf{j}^-} = C_\mathbf{j}^-, \qquad C_\mathbf{j}^+ \equiv v_\mathbf{j} \bmod q\mathscr{L}_\mathbf{i}^+, \qquad C_\mathbf{j}^- \equiv v_\mathbf{j} \bmod q^{-1}\mathscr{L}_\mathbf{i}^-,$$

where $x \mapsto \bar{x}$ denotes the bar involution of $\widehat{H}_r u_\mathbf{i}$ defined in §3.4. By collecting them together for all $\mathbf{i} \in A_r$, we obtain two bases $C_\mathbf{j}^+$, $C_\mathbf{j}^-$ $(\mathbf{j} \in \mathbb{Z}^r)$ of $U^{\otimes r}$ called the Kazhdan-Lusztig bases. The parabolic Kazhdan-Lusztig polynomials are then defined via the expansions

$$C_\mathbf{j}^+ = \sum_{\mathbf{k} \in \widehat{\mathfrak{S}}_r \mathbf{i}} P_{\mathbf{k},\mathbf{j}}^+(q)\, v_\mathbf{k}, \qquad C_\mathbf{j}^- = \sum_{\mathbf{k} \in \widehat{\mathfrak{S}}_r \mathbf{i}} P_{\mathbf{k},\mathbf{j}}^-(-q^{-1})\, v_\mathbf{k}.$$

They can be calculated inductively, as illustrated by the next example.

Example 4 Let us take $r = 3$, $n = 2$ and compute $C^-_{\mathbf{j}}$ for $\mathbf{j} = (0,6,1)$. We have $(0,6,1) \in \widehat{\mathfrak{S}}_3\mathbf{i}$ with $\mathbf{i} = (1,2,2) \in A_3$. So we can start with $C^-_{(1,2,2)} = v_{(1,2,2)}$. Next we calculate that

$$(0,6,1) = s_2 s_0 s_1 s_2 s_0 \tau \cdot (1,2,2).$$

Clearly

$$C^-_{(2,2,3)} = C^-_{\tau(1,2,2)} = v_{(2,2,3)}.$$

Now note that for every $i = 0,1,2$ we have $\overline{T_i - q^{-1}} = T_i - q^{-1}$, hence applying $T_i - q^{-1}$ to a Kazhdan-Lusztig element $C^-_{\mathbf{k}}$ gives a bar-invariant element. Then, setting for short $t = q^{-1}$, we compute successively

$$\begin{aligned}
(T_0 - t)v_{(2,2,3)} &= v_{(1,2,4)} - tv_{(2,2,3)} = C^-_{(1,2,4)},\\
(T_2 - t)C^-_{(1,2,4)} &= v_{(1,4,2)} - tv_{(1,2,4)} - tv_{(2,3,2)} + t^2 v_{(2,2,3)} = C^-_{(1,4,2)},\\
(T_1 - t)C^-_{(1,4,2)} &= v_{(4,1,2)} - tv_{(1,4,2)} - tv_{(2,1,4)} + t^2 v_{(1,2,4)} - tv_{(3,2,2)} + t^2 v_{(2,3,2)} = C^-_{(4,1,2)},\\
(T_0 - t)C^-_{(4,1,2)} &= v_{(0,1,6)} - tv_{(4,1,2)} - tv_{(0,4,3)} + t^2 v_{(1,4,2)} + t^2 v_{(2,2,3)} - tv_{(1,2,4)}\\
&\qquad - tv_{(0,2,5)} + t^2 v_{(3,2,2)} + t^2 v_{(0,3,4)} - t^3 v_{(2,3,2)} = C^-_{(0,1,6)},\\
(T_2 - t)C^-_{(0,1,6)} &= v_{(0,6,1)} - tv_{(0,1,6)} - tv_{(4,2,1)} + t^2 v_{(4,1,2)} - tv_{(0,3,4)}\\
&\qquad + v_{(0,4,3)} + 2t^2 v_{(1,2,4)} - 2tv_{(1,4,2)} + 2t^2 v_{(2,3,2)} - 2t^3 v_{(2,2,3)}\\
&\qquad - tv_{(0,5,2)} + t^2 v_{(0,2,5)} + t^2 v_{(0,4,3)} - t^3 v_{(0,3,4)}.
\end{aligned}$$

We see that this last vector $v \equiv v_{(0,6,1)} + v_{(0,4,3)} \mod t\mathscr{L}^-_{\mathbf{i}}$. Thus, subtracting

$$C^-_{(0,4,3)} = v_{(0,4,3)} - tv_{(0,3,4)} - tv_{(1,4,2)} + t^2 v_{(1,2,4)} + t^2 v_{(2,3,2)} - t^3 v_{(2,2,3)},$$

which we can assume already calculated by induction, we get

$$\begin{aligned}
C^-_{(0,6,1)} &= v_{(0,6,1)} - tv_{(0,1,6)} - tv_{(4,2,1)} + t^2 v_{(4,1,2)} + t^2 v_{(1,2,4)} - tv_{(1,4,2)}\\
&\quad + t^2 v_{(2,3,2)} - t^3 v_{(2,2,3)} - tv_{(0,5,2)} + t^2 v_{(0,2,5)} + t^2 v_{(0,4,3)} - t^3 v_{(0,3,4)}.
\end{aligned}$$

4.3 Categorification of $\wedge^r U$

We can now relate the canonical bases of $\wedge^r_q U$ to the representation theory of $U_\zeta(\mathfrak{gl}_r)$. Recall the polynomials $l_{\mathbf{ij}}(q)$ defined in §3.7.

Theorem 4 (Varagnolo-Vasserot [VV]) *For* $\mathbf{i},\mathbf{j} \in \mathbb{Z}^r_>$, *we have* $l_{\mathbf{i},\mathbf{j}} = P^-_{\mathbf{j},\mathbf{i}}$.

Proof — Take $\mathbf{i} \in \mathbb{Z}^r_>$ and consider the element $D_{\mathbf{i}} := \mathrm{pr}(C^-_{\mathbf{i}}) \in \wedge^r_q U$. Then $\overline{D_{\mathbf{i}}} = D_{\mathbf{i}}$ by definition of the bar involution on $\wedge^r_q U$. Note that if $\mathbf{j} \in \mathbb{Z}^r_>$ then $\mathrm{pr}(v_{\mathbf{j}}) = \mathrm{pr}(u_{\mathbf{j}}) = q^{l(w_0)} \wedge_q u_{\mathbf{j}}$ by Lemma 2. Otherwise, if $j_k < j_{k+1}$ then $\mathrm{pr}(v_{\mathbf{j}}) = -q^{-1}\mathrm{pr}(v_{s_k\cdot\mathbf{j}})$, as follows immediately from the definition of pr and Eq. (57). Therefore if $\mathbf{k} \in \mathbb{Z}^r_>$ and $s \in \mathfrak{S}_r$ we have $\mathrm{pr}(v_{s\cdot\mathbf{k}}) = (-q)^{-l(s)} q^{l(w_0)} \wedge_q u_{\mathbf{k}}$. We now use the following simple observation (see [**So**, Remark 3.2.4]): if $i_k > i_{k+1}$ and $j_k > j_{k+1}$ then $P^-_{s_k\cdot\mathbf{j},\mathbf{i}}(-q^{-1}) = -q^{-1}P^-_{\mathbf{j},\mathbf{i}}(-q^{-1})$. Since $\mathbf{i} \in \mathbb{Z}^r_>$, this implies that

$$D_{\mathbf{i}} = [r]! \sum_{\mathbf{j}\in\mathbb{Z}^r_>} P^-_{\mathbf{j},\mathbf{i}}(-q^{-1}) \wedge_q u_{\mathbf{j}},$$

where

$$[r]! = [r][r-1]\cdots[1] = q^{l(w_0)} \sum_{w\in\mathfrak{S}_r} q^{-2l(w)}$$

is bar-invariant. Hence $(1/[r]!)D_{\mathbf{i}}$ is bar-invariant and congruent to $\wedge_q u_{\mathbf{i}}$ modulo $q^{-1}L^-$. Thus $D_{\mathbf{i}} = [r]!\, G^-_{\mathbf{i}}$ and the theorem is proved. □

This theorem has the following nice reformulation. Define a $\mathbb{Z}$-linear map ι from the Grothendieck group of finite-dimensional representations of $U_\zeta(\mathfrak{gl}_r)$ to $\wedge^r U$ by

$$\iota[W_\zeta(\lambda)] = \wedge u_{\lambda+\rho}, \qquad (\lambda \in \mathbb{Z}^r_{\geqslant}).$$

Then comparing Theorem 4 and the Lusztig character formula (43) we see that $\iota[L(\lambda)]$ is equal to the specialization at $q=1$ of $G^-_{\lambda+\rho}$. In other words, using a more fancy language, the category of finite-dimensional representations of $U_\zeta(\mathfrak{gl}_r)$ is a *categorification* of the vector space $\wedge^r U$ endowed with the specialization at $q=1$ of the canonical basis $G^-_{\mathbf{i}}$ $(\mathbf{i} \in \mathbb{Z}^r_>)$.

Remark 3 The above proof of the relation between the canonical basis $G^-_{\mathbf{i}}$ $(\mathbf{i} \in \mathbb{Z}^r_>)$ and the simple $U_\zeta(\mathfrak{gl}_r)$-modules relies on Lusztig's character formula, whose proof requires a lot of work. There are now two other proofs which do not use Lusztig's formula, and therefore provide an independent proof of this formula (for type A). The first one is due to Varagnolo-Vasserot-Schiffmann [**VV**, **Sc**]. It relies on a geometric construction of the affine v-Schur algebra. The second one [**L1**] uses Ariki's theorem [**A1**]. Ariki's theorem gives a proof of the LLT-conjecture [**LLT**], which relates the simple modules of the finite Hecke algebras $H_r(\zeta)$ with a subset of the canonical basis $G^+_{\mathbf{i}}$ $(\mathbf{i} \in \mathbb{Z}^r_>)$. In [**L1**] it is proved that Ariki's theorem implies the Lusztig character formula (for type A).

4.4 Categorification of the Fock space

We first describe an interesting symmetry of the bar involution of the Fock space $\mathscr{F}$. Define a scalar product on $\mathscr{F}$ by

$$\langle |\lambda\rangle, |\mu\rangle\rangle = \delta_{\lambda\mu}, \qquad (\lambda,\mu \in \mathscr{P}).$$

Define also a semi-linear involution $v \mapsto v'$ on $\mathscr{F}$ by setting

$$q' = q^{-1}, \qquad |\lambda\rangle' = |\lambda'\rangle,$$

where λ' denotes the partition conjugate to $\lambda \in \mathscr{P}$ (that is, the rows of the Young diagram of λ' are the columns of the Young diagram of λ). One can show [**LT2**, Th. 7.11] that

$$\langle \overline{u}, v\rangle = \langle u', \overline{v'}\rangle, \qquad (u,v \in \mathscr{F}). \tag{58}$$

Let $\{G^*_\lambda\}$ denote the basis of $\mathscr{F}$ adjoint to $\{G^+_\lambda\}$ for the above scalar product. In other words, $\langle G^*_\lambda, G^+_\mu\rangle = \delta_{\lambda,\mu}$. Write

$$G^*_\lambda = \sum_\beta g_{\lambda,\mu}(q)\,|\lambda\rangle, \qquad \text{and} \qquad \mathbf{G}_k := [g_{\lambda,\mu}(q)], \qquad (\lambda,\mu \vdash k).$$

Since $\{|\lambda\rangle\}$ is an orthonormal basis, we have $\mathbf{G}_k = \mathbf{D}_k^{-1}$, a unitriangular matrix with off-diagonal entries in $q\mathbb{Z}[q]$.

Proposition 15 *For $\lambda \in \mathscr{P}$ one has $(G^*_\lambda)' = G^-_{\lambda'}$.*

Proof — We have to prove that $(G^*_\lambda)'$ satisfies the two defining properties of $G^-_{\lambda'}$, namely

$$(G^*_\lambda)' \equiv |\lambda'\rangle \bmod q^{-1}L^-_\infty, \qquad \overline{(G^*_\lambda)'} = (G^*_\lambda)'.$$

Since $\mathbf{G}_k$ is unitriangular with off-diagonal entries in $q\mathbb{Z}[q]$, $G^*_\lambda \equiv |\lambda\rangle \bmod qL^+_\infty$, which implies that $(G^*_\lambda)' \equiv |\lambda'\rangle \bmod q^{-1}L^-_\infty$. The second property is equivalent to

$$\langle \overline{(G^*_\lambda)'}\,,\,(G^+_\mu)'\rangle = \delta_{\lambda,\mu}, \qquad (\lambda,\ \mu \vdash k),$$

because $\{(G^*_\lambda)'\}$ is the basis adjoint to $\{(G^+_\mu)'\}$. Now, by Eq. (58),

$$\langle \overline{(G^*_\lambda)'}\,,\,(G^+_\mu)'\rangle = \langle G^*_\lambda\,,\,\overline{G^+_\mu}\rangle = \langle G^*_\lambda\,,\,G^+_\mu\rangle = \delta_{\lambda,\mu}.$$

□

Proposition 15 amounts to say that $g_{\lambda\mu}(q) = e_{\lambda'\mu'}(q)$, or in other words that

$$\sum_{\gamma\vdash k} e_{\lambda'\gamma'}(-q)\, d_{\gamma\mu}(q) = \delta_{\lambda\mu}, \qquad (\lambda,\mu \vdash k).$$

In particular, setting $q=1$ we get

$$\sum_{\gamma\vdash k} e_{\lambda'\gamma'}(-1)\, d_{\gamma\mu}(1) = \delta_{\lambda\mu}, \qquad (\lambda,\mu \vdash k).$$

Since by Theorem 4, we have that the coefficient of $\mathrm{ch}W_\zeta(\mu)$ in $\mathrm{ch}L(\lambda)$ is equal to

$$l_{\lambda+\rho,\mu+\rho}(-1) = e_{\lambda\mu}(-1),$$

it follows that

$$d_{\lambda\mu}(1) = [W_\zeta(\lambda')\,:\,L(\mu')]$$

is the multiplicity of $L(\mu')$ as a composition factor of the Weyl module $W_\zeta(\lambda')$. Thus the values at $q=1$ of the coefficients $d_{\lambda\mu}(q)$ of G^+_μ are decomposition numbers for $U_\zeta(\mathfrak{gl}_k)$, as was conjectured in [**LT1**].

It is more natural to state these results in terms of the v-Schur algebras $\mathscr{S}_k(\xi)$. Here $\xi = \zeta^2$ and, by definition, $\mathscr{S}_k(\xi)$ is the image of $U_\zeta(\mathfrak{gl}_k)$ in the endomorphism ring of the kth tensor power of its defining k-dimensional representation. Thus, the category of $\mathscr{S}_k(\xi)$-modules is nothing else than the category of polynomial representations of degree k of $U_\zeta(\mathfrak{gl}_k)$. The algebra $\mathscr{S}_k(\xi)$ is quasi-hereditary, and its simple objects can be identified with the $L(\lambda)$ for $\lambda \vdash k$. (For a nice exposition of the theory of quasi-hereditary algebras and their tilting modules, we refer to [**DR**]). Consider the category

$$\mathscr{C} = \bigoplus_{k\in\mathbb{N}} \mathrm{mod}\,\mathscr{S}_k(\xi).$$

This can be regarded as a categorification of the Fock space $\mathscr{F}$, with the G^-_λ being the classes of the simple objects $L(\lambda)$ of $\mathscr{C}$, and the $|\lambda\rangle$ the classes of the standard objects $W_\zeta(\lambda)$.

Moreover, let $T(\lambda)$ denote the indecomposable tilting $\mathscr{S}_k(\xi)$-module with highest weight λ. By [**DPS**, Prop. 8.2] which states that

$$[W_\zeta(\lambda') : L(\mu')] = [T(\mu) : W_\zeta(\lambda)],$$

we see that $[T(\mu) : W_\zeta(\lambda)] = d_{\lambda,\mu}(1)$, and the $G^+(\lambda)$ are the classes of the indecomposable tilting objects $T(\lambda)$.

5 Fock space representations of $U_q(\widehat{\mathfrak{sl}}_n)$: higher level

In this section we sketch following Uglov [**TU**, **U1**, **U2**] a q-analogue of the constructions of §2.3. We fix an integer $\ell \geqslant 1$.

5.1 Action on U

The q-analogue of the action of $\widehat{\mathfrak{g}}$ on $V(z)$ used in §2.3.2 (a) is the $\mathbb{Q}(q)$-vector space

$$U := \bigoplus_{k\in\mathbb{Z}} \mathbb{Q}(q) u_k,$$

with the $U_q(\widehat{\mathfrak{g}})$-action given by

$$\begin{array}{rclrcll} e_i u_k &=& \delta_{k\equiv i+1}\, u_{k-1}, & f_i u_k &=& \delta_{k\equiv i}\, u_{k+1}, & (1 \leqslant i \leqslant n-1), \\ e_0 u_k &=& \delta_{k\equiv 1}\, u_{k-1-(\ell-1)n}, & f_0 u_k &=& \delta_{k\equiv 0}\, u_{k+1+(\ell-1)n}, & \\ t_i u_k &=& q^{\delta_{k\equiv i}-\delta_{k\equiv i+1}}\, u_k, & & & & (0 \leqslant i \leqslant n-1). \end{array}$$

Exercise 18 Check that this defines a level 0 representation of $U_q(\widehat{\mathfrak{g}})$.

We consider again

$$V = \bigoplus_{i=1}^{n} \mathbb{Q}(q) v_i,$$

and also

$$W = \bigoplus_{j=1}^{\ell} \mathbb{Q}(q) w_j.$$

Then we can identify $W \otimes V \otimes \mathbb{Q}(q)[z,z^{-1}]$ with U by

$$w_j \otimes v_i \otimes z^k \equiv u_{i+(j-1)n-\ell nk}, \qquad (1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant \ell,\ k \in \mathbb{Z}). \tag{59}$$

The above action can be rewritten as

$$\begin{array}{rcll} e_i(w_d \otimes v_c \otimes z^m) &=& \delta_{(c\equiv i+1 \bmod n)}\ w_d \otimes v_{c-1} \otimes z^{m+\delta_{i,0}}, & (0 \leqslant i \leqslant n-1), \\ f_i(w_d \otimes v_c \otimes z^m) &=& \delta_{(c\equiv i \bmod n)}\ w_d \otimes v_{c+1} \otimes z^{m-\delta_{i,0}}, & (0 \leqslant i \leqslant n-1), \\ t_i(w_d \otimes v_c \otimes z^m) &=& q^{(\delta_{c\equiv i \bmod n}-\delta_{c\equiv i+1 \bmod n})}\ w_d \otimes v_c \otimes z^m, & (0 \leqslant i \leqslant n-1). \end{array}$$

Here it is understood that $v_0 = v_n$ and $v_{n+1} = v_1$.

Put $p := -q^{-1}$ and consider the quantum affine algebra $U_p(\widetilde{\mathfrak{g}}) = U_p(\widehat{\mathfrak{sl}}_\ell)$. To avoid confusion, we denote its generators by $\dot{e}_i, \dot{f}_i, \dot{t}_i$ $(0 \leqslant i \leqslant \ell-1)$. It acts on U by

$$\begin{array}{rcll} \dot{e}_i(w_d \otimes v_c \otimes z^m) &=& \delta_{(d\equiv i+1 \bmod \ell)}\ w_{d-1} \otimes v_c \otimes z^{m+\delta_{i,0}}, & (0 \leqslant i \leqslant \ell-1), \\ \dot{f}_i(w_d \otimes v_c \otimes z^m) &=& \delta_{(d\equiv i \bmod \ell)}\ w_{d+1} \otimes v_c \otimes z^{m-\delta_{i,0}}, & (0 \leqslant i \leqslant \ell-1), \\ \dot{t}_i(w_d \otimes v_c \otimes z^m) &=& p^{(\delta_{d\equiv i \bmod \ell}-\delta_{d\equiv i+1 \bmod \ell})}\ w_d \otimes v_c \otimes z^m, & (0 \leqslant i \leqslant \ell-1). \end{array}$$

Here it is understood that $w_0 = w_\ell$ and $w_{\ell+1} = w_1$. Clearly, the actions of $U_q(\widehat{\mathfrak{g}})$ and $U_p(\widetilde{\mathfrak{g}})$ commute with each other.

5.2 The tensor spaces $U^{\otimes r}$

Let $r \geqslant 1$. Using the comultiplication Δ of (22) we get a level 0 action of $U_q(\widehat{\mathfrak{g}})$ on $U^{\otimes r}$. Similarly, we endow $U^{\otimes r}$ with a level 0 action of $U_p(\widetilde{\mathfrak{g}})$ using the comultiplication

$$\Delta\dot{f}_i = \dot{f}_i \otimes 1 + \dot{t}_i \otimes \dot{f}_i, \quad \Delta\dot{e}_i = \dot{e}_i \otimes \dot{t}_i^{-1} + 1 \otimes \dot{e}_i, \quad \Delta\dot{t}_i^{\pm} = \dot{t}_i^{\pm} \otimes \dot{t}_i^{\pm}. \tag{60}$$

We have endowed $V^{\otimes r} \otimes \mathbb{Q}(q)[z_1^{\pm}, \ldots, z_r^{\pm}]$ with a left action of $\widehat{H}_r$ in §3.4. We can also define a right action of H_r on $W^{\otimes r}$ by

$$w_{\mathbf{i}} T_k = \begin{cases} -w_{s_k \cdot \mathbf{i}} & \text{if } i_k < i_{k+1}, \\ (-q) w_{\mathbf{i}} & \text{if } i_k = i_{k+1}, \\ -w_{s_k \cdot \mathbf{i}} + (q^{-1} - q) w_{\mathbf{i}} & \text{if } i_k > i_{k+1}, \end{cases} \qquad (1 \leqslant k \leqslant r-1).$$

Here, for $\mathbf{i} = (i_1, \ldots, i_\ell) \in [1, \ell]^r$ we write $w_{\mathbf{i}} = w_{i_1} \otimes \cdots \otimes w_{i_\ell} \in W^{\otimes r}$. This is a q-analogue of the right action of $\mathfrak{S}_r$ on $W^{\otimes r}$ given in §2.3.2 (b).

5.3 The q-wedge spaces $\wedge_q^r U$

Following Uglov, we can now define

$$\wedge_q^r U := W^{\otimes r} \otimes_{H_r} \left(V^{\otimes r} \otimes \mathbb{Q}(q)[z_1^{\pm}, \ldots, z_r^{\pm}] \right). \tag{61}$$

Note that this is indeed a q-analogue of $\wedge^r U$, by Proposition 3. As in the case $\ell = 1$, the q-wedge space $\wedge_q^r U$ is endowed by construction with a basis of normally ordered q-wedges $\{\wedge_q u_{\mathbf{i}} \mid \mathbf{i} \in \mathbb{Z}^r\}$. However, the straightening relations are now significantly more complicated, and they depend both on n and ℓ. We shall not reproduce them here (see *e.g.* [**U1**, §2.1]).

5.4 The Fock spaces $\mathbf{F}[\mathbf{m}_\ell]$

Passing to the limit $r \to \infty$, we get for every $m \in \mathbb{Z}$ a $\mathbb{Q}(q)$-vector space $\mathscr{F}_m[\ell]$ with a *standard basis* consisting of all infinite q-wedges

$$\wedge_q u_{\mathbf{i}} := u_{i_1} \wedge_q u_{i_2} \wedge_q \cdots \wedge_q u_{i_r} \wedge_q \cdots, \qquad (\mathbf{i} = (i_1 > i_2 > \cdots > i_r > \cdots) \in \mathbb{Z}^{\mathbb{N}^*}),$$

which coincide, except for finitely many indices, with

$$|\emptyset_m\rangle := u_m \wedge_q u_{m-1} \wedge_q \cdots \wedge_q u_{m-r} \wedge_q u_{m-r-1} \wedge_q \cdots$$

We can label the elements of this basis by partitions in exactly the same way as in §2.3.3. We shall write, with the same notation

$$\wedge_q u_{\mathbf{i}} = |\lambda, m\rangle = |\underline{\lambda}_\ell, \mathbf{m}_\ell\rangle.$$

We can then define $\mathbf{F}[\mathbf{m}_\ell]$ as the subspace of $\mathscr{F}_m[\ell]$ spanned by all vectors of the standard basis of the form $|\underline{\lambda}_\ell, \mathbf{m}_\ell\rangle$ for some ℓ-tuple of partitions $\underline{\lambda}_\ell$. This is the *q-deformed Fock space of level ℓ with multi-charge* $\mathbf{m}_\ell$. Indeed, it is a level ℓ representation of $U_q(\widehat{\mathfrak{g}})$. The isomorphism type of $\mathbf{F}[\mathbf{m}_\ell]$ as a $U_q(\widehat{\mathfrak{g}})$-module depends only on the (unordered) list of residues modulo n of the components of $\mathbf{m}_\ell$. There are some nice q-analogues of the combinatorial formulas of Proposition 4 for the action of the Chevalley generators on the standard basis (see *e.g.* [**U2**, §2.1]).

The Fock space $\mathbf{F}[\mathbf{m}_\ell]$ is also endowed with the action of bosons B_k given by

$$B_k(\wedge_q u_{\mathbf{i}}) = \sum_{j \geqslant 1} \wedge_q u_{\mathbf{i} - n\ell k \varepsilon_j}, \qquad (k \in \mathbb{Z}^*). \tag{62}$$

They generate a Heisenberg algebra [**U2**]:

$$[B_k, B_m] = \delta_{k,-m}\, k\, \frac{(1-q^{-2kn})(1-q^{-2k\ell})}{(1-q^{-2k})^2}, \qquad (k,m \in \mathbb{Z}^*).$$

5.5 The canonical bases of $\mathbf{F}[\mathbf{m}_\ell]$

One can define a bar involution on $\mathscr{F}_m[\ell]$, either by flipping q-wedges (as in Prop. 10) or by using the bar involution of $\widehat{H}_r$. It is expressed by a unitriangular matrix on the standard basis, and it preserves the subspaces $\mathbf{F}[\mathbf{m}_\ell]$. This allows to introduce, as in §3.13, canonical bases $\{G(\underline{\lambda}_\ell, \mathbf{m}_\ell)^+ \,|\, \underline{\lambda}_\ell \in \mathscr{P}^\ell\}$ and $\{G(\underline{\lambda}_\ell, \mathbf{m}_\ell)^- \,|\, \underline{\lambda}_\ell \in \mathscr{P}^\ell\}$ of $\mathbf{F}[\mathbf{m}_\ell]$ characterized by:

(i) $\overline{G(\underline{\lambda}_\ell, \mathbf{m}_\ell)^+} = G(\underline{\lambda}_\ell, \mathbf{m}_\ell)^+, \quad \overline{G(\underline{\lambda}_\ell, \mathbf{m}_\ell)^-} = G(\underline{\lambda}_\ell, \mathbf{m}_\ell)^-,$

(ii) $G(\underline{\lambda}_\ell, \mathbf{m}_\ell)^+ \equiv |\underline{\lambda}_\ell, \mathbf{m}_\ell\rangle \bmod qL^+[\mathbf{m}_\ell], \quad G(\underline{\lambda}_\ell, \mathbf{m}_\ell)^- \equiv |\underline{\lambda}_\ell, \mathbf{m}_\ell\rangle \bmod q^{-1}L^-[\mathbf{m}_\ell],$

where $L^+[\mathbf{m}_\ell]$ (*resp.* $L^-[\mathbf{m}_\ell]$) is the $\mathbb{Z}[q]$-submodule (*resp.* $\mathbb{Z}[q^{-1}]$-submodule) of $\mathbf{F}[\mathbf{m}_\ell]$ spanned by the elements of the standard basis.

We denote by $\Delta_k^\pm[\mathbf{m}_\ell]$ the transition matrix from the standard basis to $\{G(\underline{\lambda}_\ell, \mathbf{m}_\ell)^\pm \,|\, \underline{\lambda}_\ell \in \mathscr{P}^\ell\}$ in the degree k component of $\mathbf{F}[\mathbf{m}_\ell]$. Uglov [**U2**, Th. 3.26] has given an expression of the entries of $\Delta_k^\pm[\mathbf{m}_\ell]$ in terms of parabolic Kazhdan-Lusztig polynomials for $\widetilde{\mathfrak{S}}_r$ which generalizes Theorem 4.

5.6 Comparison of bases

Let $\mathbf{m}_\ell = (m_1, \ldots, m_\ell)$ and $\mathbf{m}'_\ell = (m'_1, \ldots, m'_\ell)$ be two multi-charges such that

$$m_i \equiv m'_i \bmod n, \qquad (1 \leqslant i \leqslant \ell). \tag{63}$$

We have seen that $\mathbf{F}[\mathbf{m}_\ell]$ and $\mathbf{F}[\mathbf{m}'_\ell]$ are isomorphic representations of $U_q(\widehat{\mathfrak{g}})$, but the formulas for the action of the Chevalley generators on the standard basis of these two Fock spaces are not the same, and the matrices $\Delta_k^\pm[\mathbf{m}_\ell]$ and $\Delta_k^\pm[\mathbf{m}'_\ell]$ are in general different. It is a very interesting problem to determine under which additional conditions on $\mathbf{m}_\ell$ and $\mathbf{m}'_\ell$ we have an equality. A first result in this direction was proved by Yvonne [**Y1**]. He showed that if $\mathbf{m}_\ell$ and $\mathbf{m}'_\ell$ are *sufficiently dominant*, *i.e.* if

$$m_1 \gg m_2 \gg \cdots \gg m_\ell, \qquad m'_1 \gg m'_2 \gg \cdots \gg m'_\ell, \tag{64}$$

then $\Delta_k^\pm[\mathbf{m}_\ell] = \Delta_k^\pm[\mathbf{m}'_\ell]$. The proof is based on the fact that the Fock spaces $\mathbf{F}[\mathbf{m}_\ell]$ and $\mathbf{F}[\mathbf{m}'_\ell]$ are weight spaces of $\mathscr{F}_m[\ell]$ for the action of $U_p(\widehat{\mathfrak{sl}}_\ell)$ [**U2**, §4.2]. When the condition (63) is fulfilled, the corresponding $\widehat{\mathfrak{sl}}_\ell$-weights Λ and Λ' are in the same orbit of the Weyl group $\widetilde{\mathfrak{S}}_\ell$. If moreover (64) holds then $\Lambda' = s_{i_t} \cdots s_{i_1}(\Lambda)$ and for every $k = 1, \ldots, t$ the weight $s_{i_k} \cdots s_{i_1}(\Lambda)$ is extremal on its i_k-string. Yvonne then proves that the canonical basis is “preserved” under such reflexions.

5.7 Cyclotomic v-Schur algebras

Dipper, James and Mathas [**DiJaMa**] have introduced some ℓ-cyclotomic analogues of the Schur algebras $\mathscr{S}_k(\xi)$ of §4.4. These are quasi-hereditary algebras $\mathscr{S}_k(\xi, \mathbf{s}_\ell)$ depending on an ℓ-tuple of parameters $\mathbf{s}_\ell = (s_1, \ldots, s_\ell) \in (\mathbb{Z}/n\mathbb{Z})^\ell$. (Here, as in §4.4, we assume that ξ is a primitive nth root of 1.) The simple $\mathscr{S}_k(\xi, \mathbf{s}_\ell)$-modules and the standard $\mathscr{S}_k(\xi, \mathbf{s}_\ell)$-modules are labelled by all ℓ-tuples of partitions $\underline{\lambda}_\ell = (\lambda^{(1)}, \ldots, \lambda^{(\ell)})$ with $\sum_i |\lambda^{(i)}| = k$. James and Mathas [**JM**] have proved a Jantzen-type sum formula for the multiplicities of simple modules in the layers of the Jantzen filtration of a standard module. This allows to calculate the decomposition matrix of $\mathscr{S}_k(\xi, \mathbf{s}_\ell)$ in small rank k, but there is no known algorithm or Kazhdan-Lusztig type formula for calculating these decomposition numbers in general.

Yvonne [**Y2**] has conjectured that the decomposition matrix of $\mathscr{S}_k(\xi, \mathbf{s}_\ell)$ is equal to the evaluation at $q = 1$ of the matrix $\Delta_k^{\pm}[\mathbf{m}_\ell]$ for any multi-charge $\mathbf{m}_\ell$ satisfying

$$m_i \equiv s_i \bmod n, \quad (i = 1, \ldots, \ell) \qquad \text{and} \qquad m_1 \gg m_2 \gg \cdots \gg m_\ell.$$

In fact, he conjectures more generally that the matrix $\Delta_k^{\pm}[\mathbf{m}_\ell]$ without evaluation of q is equal to the matrix of graded decomposition numbers given by the Jantzen filtration. He then proves that this last conjecture is compatible with the James-Mathas sum formula, thus obtaining a strong support for his conjectures.

Rouquier [**R**] conjectures that all the matrices $\Delta_k^{\pm}[\mathbf{m}_\ell]$ (without any dominance assumption on $\mathbf{m}_\ell$) should have a similar interpretation in terms of some more general cyclotomic v-Schur algebras coming from rational Cherednik algebras via the Knizhnik-Zamolodchikov functor.

6 Notes

§2 : The main references are [**K**], [**F1**], [**F2**]. See [**MJD**] for the relations with soliton equations.

§3 : The main references are [**KMS**], [**LT1**], [**LT2**], [**VV**]. See [**A2**] for a more combinatorial presentation following [**MM**]. See [**L2**] for the bosonic side and the connections with symmetric functions. For a beautiful introduction to quantum groups from the viewpoint of mathematical physics, see [**J2**]. Kashiwara has written several surveys of the theory of crystal bases [**Kas3**, **Kas4**].

§4 : See [**M**] for the q-Schur algebras, their representation theory and decomposition numbers. See [**GS**] for a recent interpretation of the canonical basis $\{G^-(\lambda)\}$ of $\mathscr{F}$ in terms of characteristic cycles of modules over the rational Cherednik algebras of type A.

§5 : The main references are [**U1**], [**U2**], [**TU**].

The first construction of $\mathbf{F}[\mathbf{m}_\ell]$ (for multi-charges $\mathbf{m}_\ell$ satisfying $0 \leqslant m_1 \leqslant \cdots \leqslant m_\ell \leqslant n-1$) was given in [**JMMO**], with emphasis on combinatorial formulas and crystal bases. See also [**FLOTW**]. By changing the multi-charge $\mathbf{m}_\ell$ (as in §5.6) we obtain several multi-partition descriptions of the crystal graph of an irreducible $U_q(\widehat{\mathfrak{g}})$-module. By Ariki's theorem [**A1**], they correspond to different parametrizations of the simple modules of the Ariki-Koike algebras. The meaning of these parametrizations was explained in [**Jac**].

Fock space representations for quantum affine algebras $U_q(\widehat{\mathfrak{g}})$ of classical type have been constructed in [**KMPY**], using the theory of perfect crystals. They may have level $\ell > 1$. Note however that for $\widehat{\mathfrak{g}} = \widehat{\mathfrak{sl}}_n$ and $\ell > 1$ these Fock space representations are irreducible as $U_q(\widehat{\mathfrak{g}}) \otimes \mathscr{H}$-modules. Hence they are *not* isomorphic to the representations $\mathbf{F}[\mathbf{m}_\ell]$ for $\ell > 1$.

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