Alain Lascoux: From geometry to combinatorics

Bernard Leclerc
Université de Caen

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In memory of Alain Lascoux (1944–2013)

It is an extremely useful thing to have knowledge of the true origins of memorable discoveries.

It is not so much that thereby history may attribute to each man his own discoveries, as that the art of making discoveries should be extended by considering noteworthy examples of it.

G. W. Leibniz

GÉOMÉTRIE ALGÉBRIQUE. — *Classes de Chern d'un produit tensoriel.* Note (*) de **Alain Lascoux,** présentée par M. Henri Cartan.

Nous donnons l'expression explicite des classes de Chern d'un produit tensoriel de deux fibrés vectoriels, ainsi que celles de la puissance extérieure et de la puissance symétrique deuxièmes.

We give the explicit formula for a tensor product of vector bundles, and for the second symmetric and exterior power.

On a souvent besoin en géométrie, par exemple pour le calcul des singularités dites de Thom-Boardman dès le deuxième ordre, de l'expression des classes de Chern d'un produit tensoriel et d'un produit symétrique. Le calcul se révèle vite impraticable par la méthode usuelle [*cf.* (5)], alors que la théorie classique des fonctions de Schur permet de le mener à bien; nous renvoyons à (1) pour les développements récents.

1. PRODUIT TENSORIEL. — Soient E un fibré vectoriel complexe de rang m et $c(E)=1+c_1(E)+\ldots+c_m(E)$ sa classe de Chern dans un anneau de cohomologie convenable. Rappelons qu'on peut décomposer formellement $c(E)$ en un produit de m facteurs : $c(E)=(1+a)(1+b)\ldots$, de sorte que les $c_i(E)$ sont les fonctions symétriques élémentaires en les $a, b\ldots$ On écrira $c(E)=\Pi_{a\in E}(1+a)$. On préfère utiliser les classes de

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Problem

Calculate the Chern classes of $E \otimes F$ in terms of the Chern classes of E and F.

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$$c(E \otimes F) = \prod_{i=1}^{m} \prod_{j=1}^{n} (1 + x_i + y_j).$$

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- Writing $y^* = (1/y_1, \ldots, 1/y_n)$, we have

$$e_k(y^*) = e_{n-k}(y) / (y_1 \cdots y_n).$$

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- For a partition $\lambda=(\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_m)\subseteq(n^m)$, put

$$\lambda^*:=(n-\lambda_m\geq n-\lambda_{m-1}\geq\cdots\geq n-\lambda_1).$$

Chern classes of a tensor product

- By Cauchy formula,

$$c_{mn}(E \otimes F) = \prod_{i,j}(x_i + y_j) = \sum_{\lambda \subseteq (n^m)} s_\lambda(x) s_{(\lambda^*)'}(y).$$

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- Same calculation gives:

$$c(E\otimes F)=\prod_{i,j}((1+x_i)+y_j)=\sum_{\lambda\subseteq(n^m)} s_\lambda(1+x)s_{(\lambda^*)'}(y),$$

where

$$s_\lambda(1+x):=s_\lambda(1+x_1,\dots,1+x_m)=\frac{a_{\lambda+\delta}(1+x_1,\dots,1+x_m)}{a_\delta(1+x_1,\dots,1+x_m)}.$$

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$$a_{\lambda+\delta}(1+x_1,\ldots,1+x_m) = \sum_{\mu\subseteq\lambda} d_{\lambda\mu}\, a_{\mu+\delta}(x_1,\ldots,x_m),$$

where:

$$d_{\lambda\mu} := \det\left(\binom{\lambda_j+m-j}{\mu_i+m-i}\right)_{1\le i,j\le m}.$$

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Theorem (Lascoux 1978)

$$c(E\otimes F) = \sum_{\mu\subseteq\lambda\subseteq(n^m)} d_{\lambda\mu}\, s_\mu(x)\, s_{(\lambda^*)'}(y).$$

Lascoux and Chern

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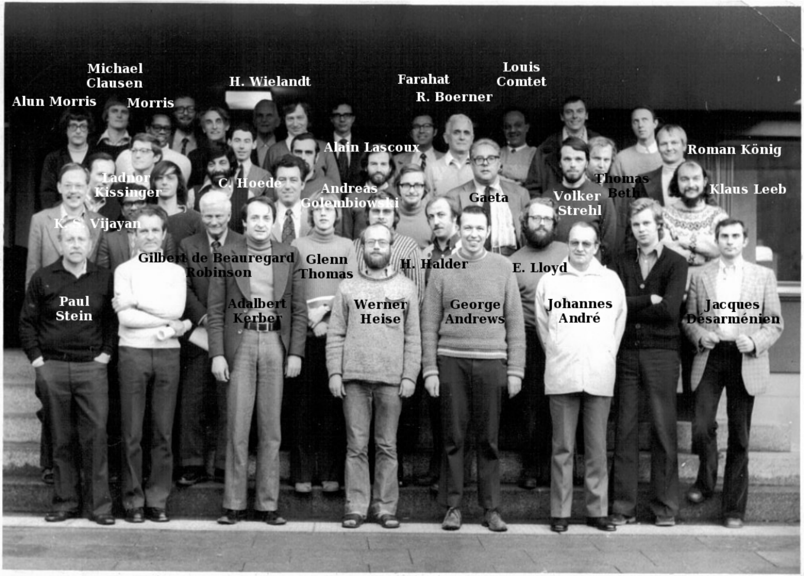
Alain Lascoux,

Combinatoire et représentation du groupe symétrique, Strasbourg 1976.


Michael Clausen
H. Wielandt
Farahat
Louis Comtet
R. Boerner
Alun Morris
Morris
Alain Lascoux
Roman König
Ladnor Kissinger
C. Hoede
Andreas Golembiowski
Gaeta
Thomas Beth
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Klaus Leeb
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Glenn Thomas
H. Halder
E. Lloyd
Paul Stein
Adalbert Kerber
Werner Heise
George Andrews
Johannes André
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- Satisfy NilHecke relations:

$$\begin{array}{rcll} \partial_i^2 & = & 0, & \\ \partial_i \partial_j & = & \partial_j \partial_i & \text{if } |i-j| \geq 1, \\ \partial_i \partial_j \partial_i & = & \partial_j \partial_i \partial_j & \text{if } |i-j| = 1. \end{array}$$

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- $y = (y_1, \ldots, y_n)$, $R := \mathbb{Z}[y]$, $R[x] = \mathbb{Z}[x, y]$.

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Definition (Lascoux-Schützenberger, 1982)

For $w \in S_n$, set

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- Simple Schubert polynomials:

$$\mathfrak{S}_w(x) := \mathfrak{S}_w(x,0).$$

Schubert polynomials ($n = 3$)

$$
\begin{tikzcd}
 & (x_1 - y_1)(x_1 - y_2)(x_2 - y_1) \arrow[dl, "\partial_1"'] \arrow[dr, "\partial_2"] & \\
(x_1 - y_1)(x_2 - y_1) \arrow[d, "\partial_2"] & & (x_1 - y_1)(x_1 - y_2) \arrow[d, "\partial_1"] \\
x_1 - y_1 \arrow[dr, "\partial_1"] & & x_1 + x_2 - y_1 - y_2 \arrow[dl, "\partial_2"] \\
 & 1 &
\end{tikzcd}
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- x-y symmetry: $\mathfrak{S}_w(y,x) = (-1)^{\ell(w)}\mathfrak{S}_{w^{-1}}(x,y)$.

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Problem

Calculate the total Chern class of the tangent bundle T of $\mathrm{Fl}(\mathbb{C}^n)$

$$c(T) = \prod_{i<j}(1 + x_i - x_j)$$

in terms of the $\mathfrak{S}_w(x)$.

GÉOMÉTRIE ALGÉBRIQUE. – *Classes de Chern des variétés de drapeaux*. Note (*) de **Alain Lascoux,** présentée par Marcel-Paul Schützenberger.

Nous étendons la formule de Cauchy (axiome n° 2, dans la numérotation de Grothendieck, de la théorie des λ-anneaux) aux polynômes de Schubert, et l'utilisons au calcul des classes de Chern des variétés de drapeaux.

ALGEBRAIC GEOMETRY. – Chern Classes of Flag Manifolds.

We extend Cauchy formula to Schubert polynomials, which are a natural generalization of Schur functions, and use it to compute the Chern classes of flag manifolds.

1. Polynômes de Schubert doubles. – On considère l'anneau commutatif $\mathbb{Z}[A]$ des polynômes en les variables de $A=\{a_1, \ldots, a_{n+1}\}$ et le groupe symétrique $W=W_{n+1}$ des permutations de A. On a défini dans [7] des opérateurs D_w, ∂_w indexés par les éléments de W; on désigne par ω l'élément de plus grande longueur de W. On pose $X_\omega=a_1^n a_2^{n-1} \ldots a_{n+1}^0$, et, suivant Demazure [3], Bernstein, Gelfand et Gelfand [1], on définit les *polynômes de Schubert* X_w par :

(1.1) $$X_w=X_\omega \partial_{\omega w}$$

(*les opérateurs sont notés à droite*).

Soit $B=\{b_1, \ldots, b_{n+1}\}$ un ensemble de même cardinal que A. On pose :

(1.2) $$\mathbb{X}_\omega(A, B)=\Pi_{i+j\leqq n+1}(a_i+b_j)$$

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we get:
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- Using Cauchy formula for expanding $\mathfrak{S}_w(x^+) = \mathfrak{S}_w(x,-1)$ yields:

Chern classes of the flag variety

Theorem (Lascoux 1982)

In the cohomology ring $\mathscr{H}$,

$$c(T) = \sum_{v,w} \mathfrak{S}_v(1)\mathfrak{S}_{vw}(x)\mathfrak{S}_{w_0 w}(x),$$

sum over $v, w \in S_n$ with $\ell(w) = \ell(v) + \ell(vw)$.

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  $\rightsquigarrow$ The $\mathfrak{S}_v(1)$ are well-understood (Macdonald's formula).
- There is no combinatorial description of the Schubert expansion of a product of Schubert polynomials.

Longueur	Permutations	c_w
0	12 345▼	1
1	12 354, 12 435▼, 13 245▼, 21 345▼	2
2	13 254, 21 354, 21 435▼	4
	12 453, 12 534, 13 425▼, 14 235▼, 23 145▼, 31 245▼	6
3	12 543, 14 325▼, 32 145▼	6
	21 453, 21 534, 23 154, 31 254	12
	14 253, 31 425▼	16
	13 524, 24 135▼	20
	13 452, 15 234, 23 415▼, 41 235▼	22
4	21 543, 32 154	12
	13 542, 15 324, 24 315▼, 42 135▼	24
	14 352, 15 243, 32 415▼, 41 325▼	28
	14 523, 34 125▼	44
	24 153, 31 524	52
	31 452, 41 253	56
	23 514, 25 134	78
	23 451, 51 234	90
5	14 523, 15 423, 34 215▼, 43 125▼	36
	15 342, 42 315▼	48
	31 542, 42 153	60
	23 541, 25 314, 52 134	102
	25 143, 32 514	104
	41 352	112
	32 451, 51 243	124
	51 324, 24 351	128
	41 523, 34 152	152
	35 124, 24 513	184
6	15 432, 43 215▼	24
	41 532, 43 152	132
	52 143, 32 541	144
	25 413, 35 214	162
	53 124, 24 531	174
	51 423, 34 251	224
	52 314, 25 341	228
	51 342, 42 351	268
	35 142, 42 513	320
	45 123, 34 512	436
7	53 214, 25 431	120
	51 432, 43 251	180
	54 123, 34 521	330
	53 142, 42 531	360
	45 213, 35 412	390
	45 132, 52 413, 43 512, 35 241	420
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8	54 213, 35 421	240
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Schubert polynomials and degeneracy loci

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- Let $h : E \to F$ be a map of vector bundles on a variety X, and

$$E_1 \subset E_2 \subset \cdots \subset E_m = E, \qquad F = F_n \to F_{n-1} \to \cdots \to F_1$$

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Theorem (Fulton, 1991)

Under appropriate conditions on the rank function $\mathbf{r}$, and for a generic map h, the class $[\Omega_{\mathbf{r}}(h)] \in H^*(X)$ is a double Schubert polynomial in the Chern roots of E and F.

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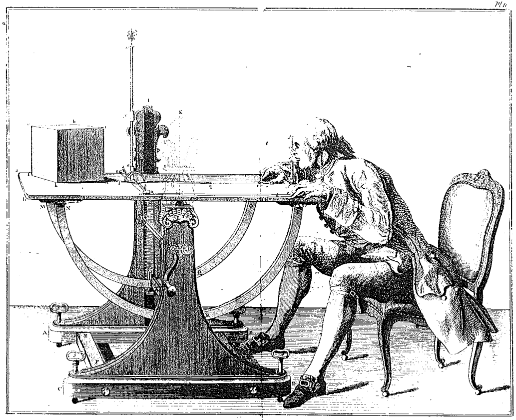
Le tirage de cette thèse a été assuré par les techniciennes de l'U.E.R. de Mathématiques de l'Université de Paris VII, Mesdames Barrier et Gilbert .



The plactic monoid

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The Foulkes conjecture

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Define the *Kostka-Foulkes* polynomials $K_{\lambda\mu}(q)$ by

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Problem (Foulkes, 1974)

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Problem (Foulkes, 1974)

Show that $K_{\lambda\mu}(q) \in \mathbb{N}[q]$ by producing a combinatorial statistics $T \mapsto \mathrm{c}(T)$ on the set of Young tableaux of shape λ and weight μ, such that:

$$K_{\lambda\mu}(q) = \sum_{T \in \mathrm{Tab}(\lambda,\mu)} q^{\mathrm{c}(T)}.$$

THÉORIE DES GROUPES. — *Sur une conjecture de* H. O. Foulkes. Note (*) de **Alain Lascoux** et **Marcel-Paul Schützenberger**, présentée par M. André Lichnerowicz.

On annonce la preuve d'une conjecture de H. O. Foulkes sur certains polynômes intervenant dans les fonctions symétriques associées aux représentations projectives du groupe symétrique et des groupes linéaires sur les corps finis.

One sketches a proof of Foulkes' conjecture on the polynomials defining Littlewood Q-functions in terms of Schur functions.

On note Z^N l'ensemble des applications I de N dans Z telles que $n\,\mathrm{I}=0$ pour tout n assez grand ce qui permet de définir $\mathrm{I}^\Sigma \in Z^N$ par $n\,\mathrm{I}^\Sigma = \sum_{m \geqq n} m\,\mathrm{I}$; le *poids* de I est donc $0\,\mathrm{I}^\Sigma$. Les partitions d'un entier n sont les $\mathrm{I} \in Z^N$ de poids n telles que $0\,\mathrm{I} \geqq 1\,\mathrm{I} \geqq 2\,\mathrm{I} \geqq \ldots$

Littlewood [1] a défini une famille basique de fonctions symétriques (en les variables d'un ensemble arbitraire qu'il est inutile d'expliciter) indexées par les partitions, $\{Q(\mathrm{I})\}$ au moyen d'une identité

(1) $$Q(\mathrm{I}) = \sum s'(\mathrm{J})\,F(\mathrm{I};\mathrm{J}),$$

dans laquelle les $s'(\mathrm{J})$ sont les fonctions de Schur (modifiées), la sommation est étendue à toutes les partitions J de même poids que I et les $F(\mathrm{I};\mathrm{J})$ sont des polynômes à coefficients entiers en une nouvelle variable q. Nous proposons d'appeler ces derniers *polynômes de Foulkes* en mémoire du regretté H. O. Foulkes auquel sont dus tant de beaux résultats sur les fonctions symétriques et qui a émis [2] la conjecture que tous leurs coefficients sont dans N. Nous faisons remarquer que ces polynômes sont les caractéristiques (polynomiales) d'Euler-Poincaré des modules inversibles de variétés drapeaux.

Nous annonçons le :

THÉORÈME I. — $F(\mathrm{I};\mathrm{J})$ *est un polynôme monique à coefficients non négatifs qui est nul si l'une des différences* $n\,\mathrm{I}^\Sigma - n\,\mathrm{J}^\Sigma$ $(n \in \mathrm{N})$ *est négative et dont le degré est égal à leur somme dans le cas*

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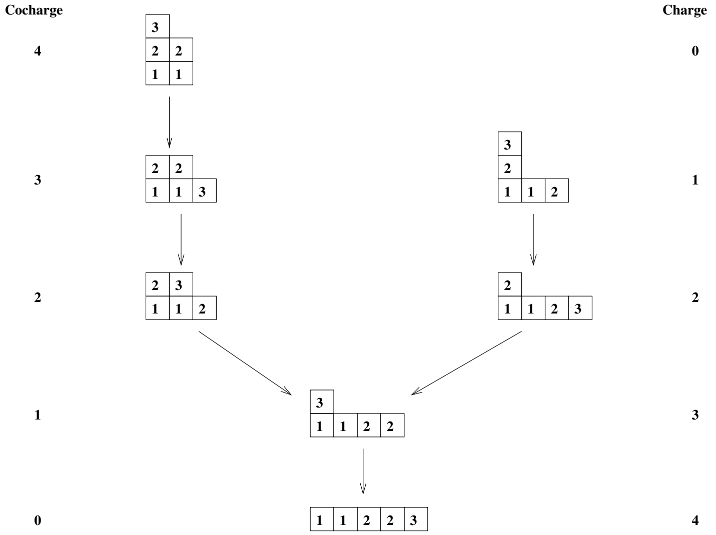
Cycle and charge ($\mu = (2,2,1)$)



The plactic monoid (continued)

Marcel-Paul Schützenberger, in *Pour le monoïde plaxique*, a letter to G.-C. Rota, 1995.

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J.-Y. Thibon and B. Leclerc are sailing up the big rivers of this continent which they are discovering. A. Lascoux is organizing the expedition, and I am watching them going off, lying in my hammock hanging from mangroves at the estuary.

LLT in action

LLT polynomials

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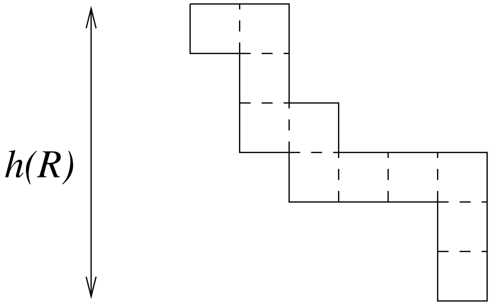
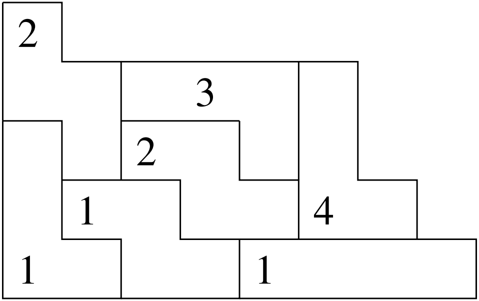
- They are defined combinatorially in terms of ribbon tableaux.

An 11-ribbon of height h(R)=6



A 4-ribbon tableau of spin 9



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- Positivity of generalized LLT $\rightsquigarrow$ new proof of the positivity of (q, t)-Kostka polynomials (Haiman-Grojnowski, 2008).

Collaborators of Alain

Alain in Poland

Le phalanstère

Alain in China

Visitor_Lascoux http://www.combinatorics.cn/people/visitor_lascoux.htm

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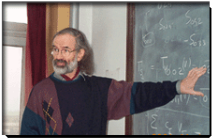

Prof. Alain Lascoux

Researcher of Centre National de la Researche Scientifique

Email:Alain.Lascoux@univ-mlv.fr
Homepage:http://www.combinatorics.net/lascoux/index.html

Research Interests

- Algebraic Combinatorics
- Symmetric Function
- Representation Theory

Teaching

- Symmetric Function

Petition for Grothendieck

PETITION

En poursuivant M.Alexandre GROTHENDIECK , mathématicien, le Parquet de Montpellier vient rappeler l'existence de l'article 21 de l'ordonnance du 2 novembre 1945, modifiée par la loi du 5 juillet 1972 :

" Tout individu qui, par aide directe ou indirecte, a facilité ou tenté de faciliter l'entrée, la circulation ou le séjour irrégulier d'un étranger est passible d'un emprisonnement de deux mois à deux ans et d'une amende de 2000 à 200.000 Francs ... "

Qui, par ailleurs, n'a contrevenu à l'article 22 de la même ordonnance :

" Toute personne logeant un étranger, en quelque qualité que ce soit, même à titre gracieux, doit en faire la déclaration dans les quarante-huit heures de l'arrivée de l'étranger, au commissaire de police ou au maire ... " ?

Dans la situation actuelle de précarité et d'insécurité des étrangers en France, qui peut se sentir à l'abri d'une menace de poursuite, pour avoir simplement voulu aider un étranger ?

En conséquence,

les soussignés protestent contre les mesures discriminatoires à l'égard des étrangers et demandent l'abrogation de l'article 21 de l'ordonnance du 2.11.45 .

NOM	QUALITE	SIGNATURE	NOM	QUALITE	SIGNATURE
CHENCINER	M.de C.		Royer	M.A	
BENNEQUIN	[illegible]		Lefèvre G.	Assist	
VOGEL	M de C		DIXMIER	professeur	
COLOMBÉ	Assist		MOEGLIN	A R CNRS	
[illegible]	[illegible]		GERARD GERGELY	M de C	
LESAFFRE	M. Assistant		HENNEÇART	AR CNRS	
CARVE	M. Assistant		STEHLE J.-L.	Chercheur Maths (I.B.M.)	
ROISIN	Assistant		HOUZEL C.	Professeur	C. Houzel
[illegible]DRAIS	Assistant		BOURGUIGNON	Chargé de recherche CNRS	
TROTMAN	Assistant		CONNES	Paris VI	
SCHREIBER	M. de Conf		Godement	Paris VII	
[illegible]	Assistant		CHAPTAL	Paris VII	
Rosenberg A.	Prof.	A Rosenberg	[illegible]	M.A.	
LE J.P.	Prof.		FLEXOR	CNRS CR	M. Flexor
STERN J.	M. Ass.		PIENE	Oslo, Norvège	Ragni Piene
MARLIN	A.R. CNRS				
ADAM	[illegible]				

Ce texte sera transmis à Grothendieck, à la presse et à différentes associations telles que l'Association française des Juristes Démocrates, le M.R.A.P. , la Ligue des Droits de l'Homme, ...

A.LASCOUX


POLYNÔMES
ALAIN LASCOUX


NANKAI UNIVERSITY